ROB 498/599: Deep Learning for Robot Perception (DeepRob)

Lecture 5: Neural Networks

https://deeprob.org/w25/

ROBOTICS

Today

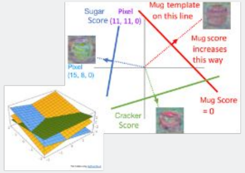
- Feedback and Recap (5min)
- Neural Networks
  - Image Features (15min)
  - Neural Networks, Activation Functions (20min)
  - Space Warping (10min)
  - Universal Approximation (10min)
  - Convex Function (10min)
- Summary and Takeaways (5min)

Recap

P1 Deadline: Feb. 2, 2025

- Use **Linear Models** for image classification problems.
- Use **Loss Functions** to express preferences over different choices of weights.
- Use **Regularization** to prevent overfitting to training data.
- Use **Stochastic Gradient Descent** to minimize our loss functions and train the model.

$$s = f(x; W) = Wx$$

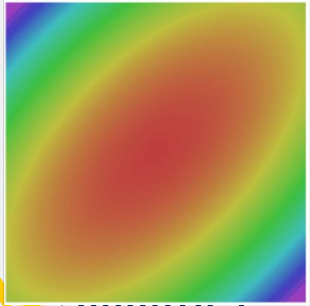


$$L_i = -\log\left(\frac{\exp^{s_{y_i}}}{\sum_i \exp^{s_j}}\right)$$ **Softmax**

$$L_i = \sum_{j \neq y_i} \max(0, s_j = - s_{y_i} + 1)$$ **SVM**

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i + R(W)$$

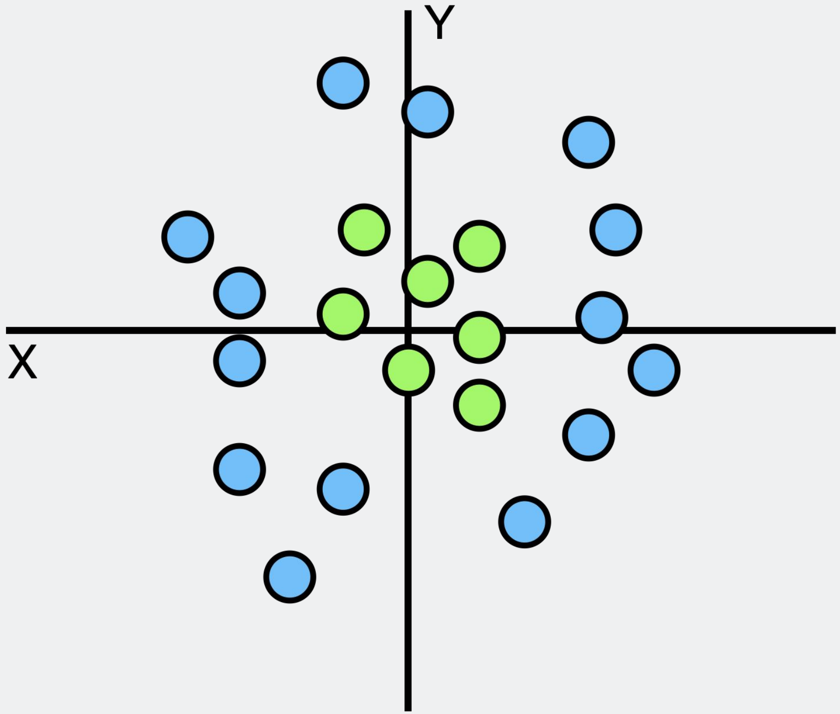
```
v = 0
for t in range(num_steps):
  dw = compute_gradient(w)
  v = rho * v + dw
  w -= learning_rate * v
```



Neural Networks

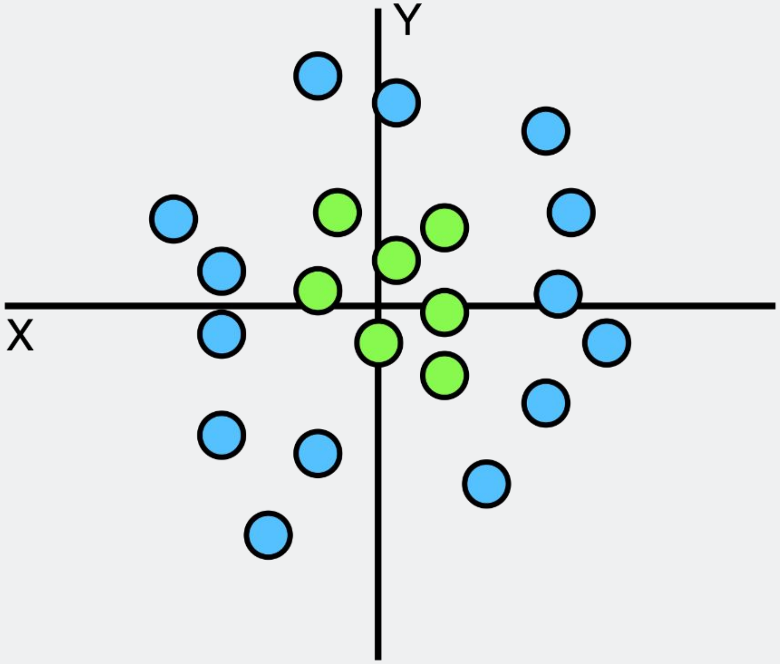
Problem: Linear Classifiers aren't that powerful

Geometric Viewpoint



Problem: Linear Classifiers aren't that powerful

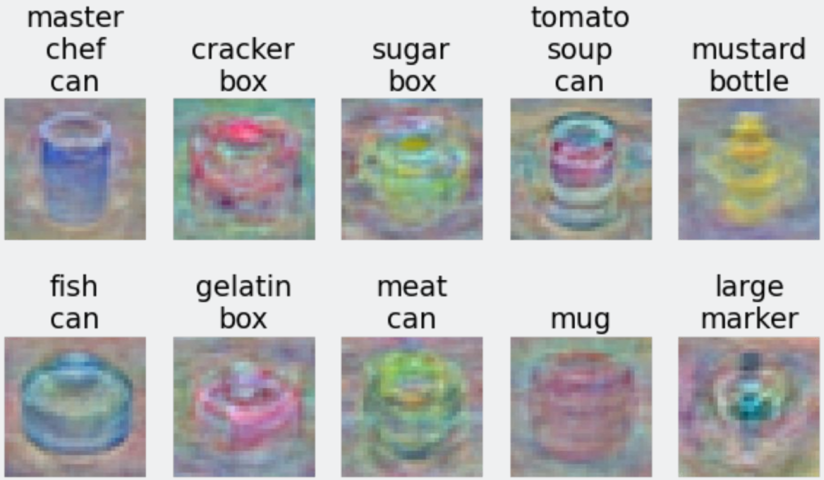
Geometric Viewpoint



Visual Viewpoint

One template per class:

Can't recognize different modes of a class



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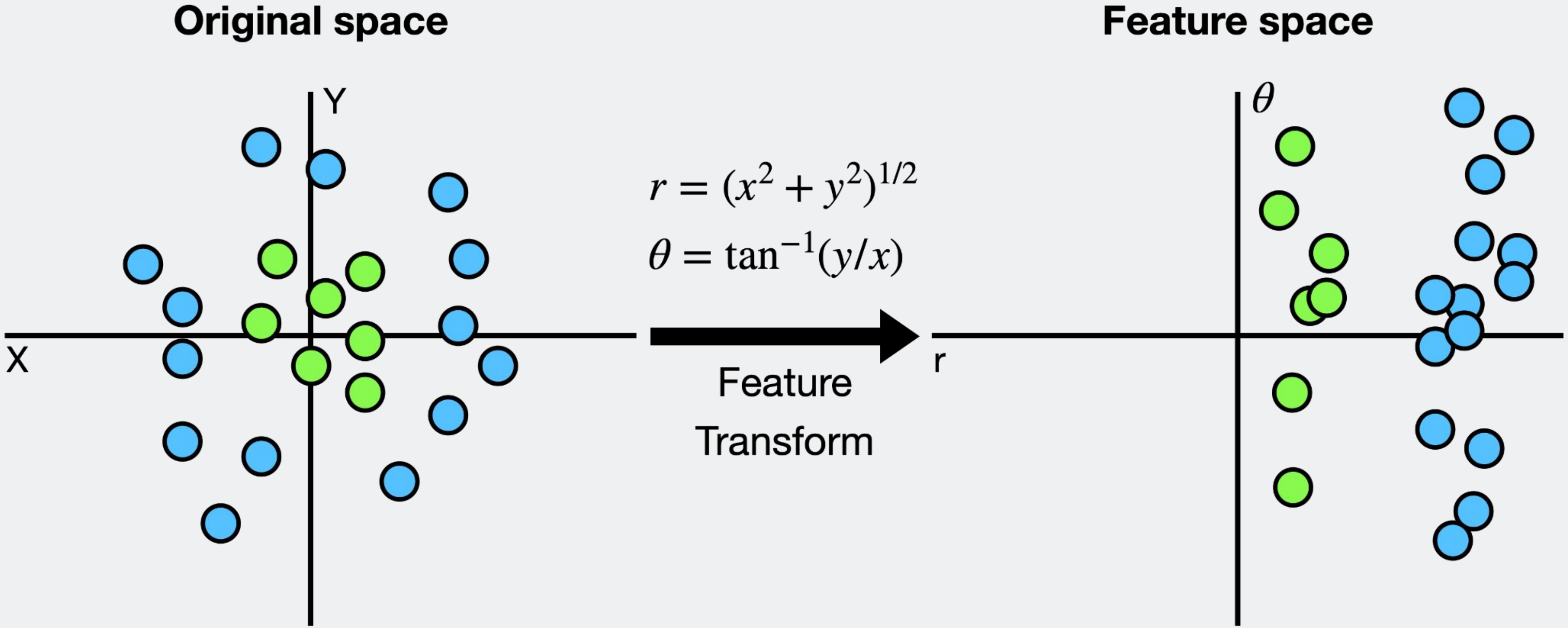
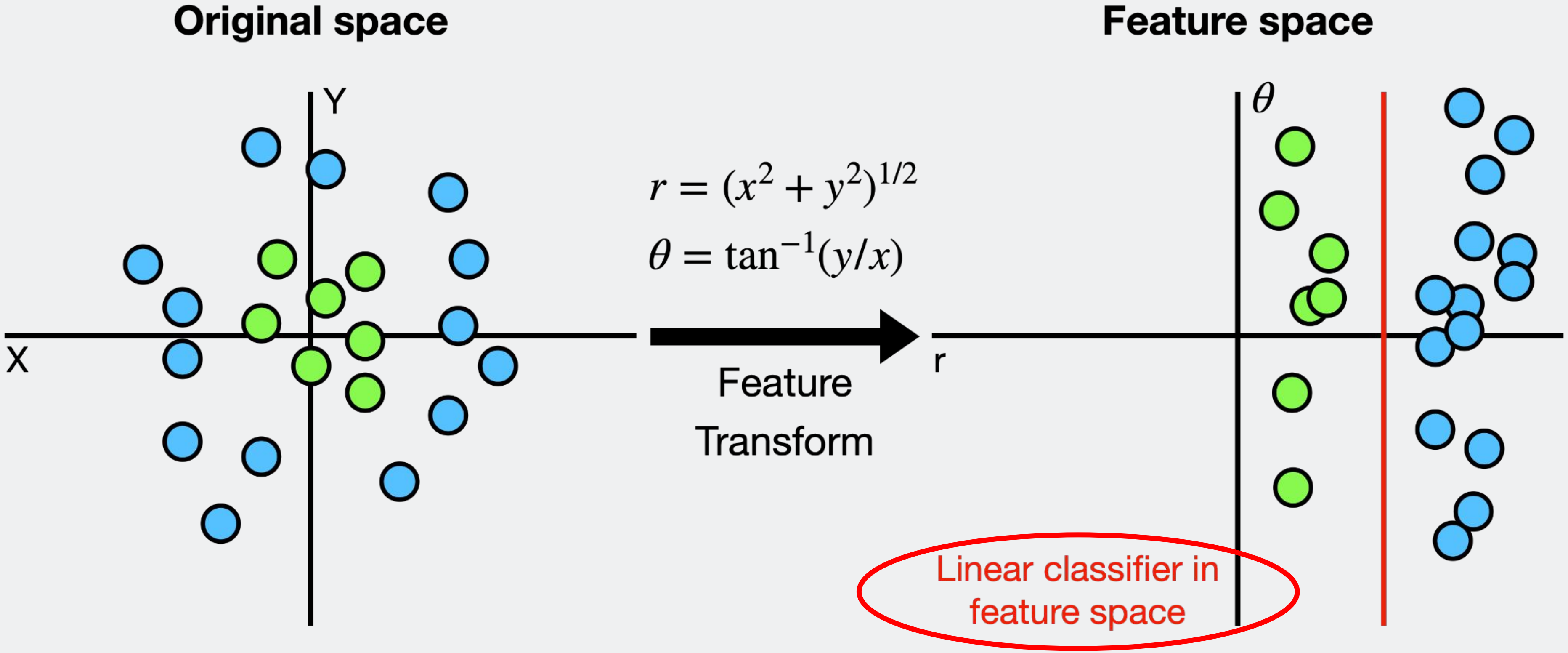
One Solution: Feature transforms



One Solution: Feature transforms



$$r = (x^2 + y^2)^{1/2}$$
$$\theta = \tan^{-1}(y/x)$$

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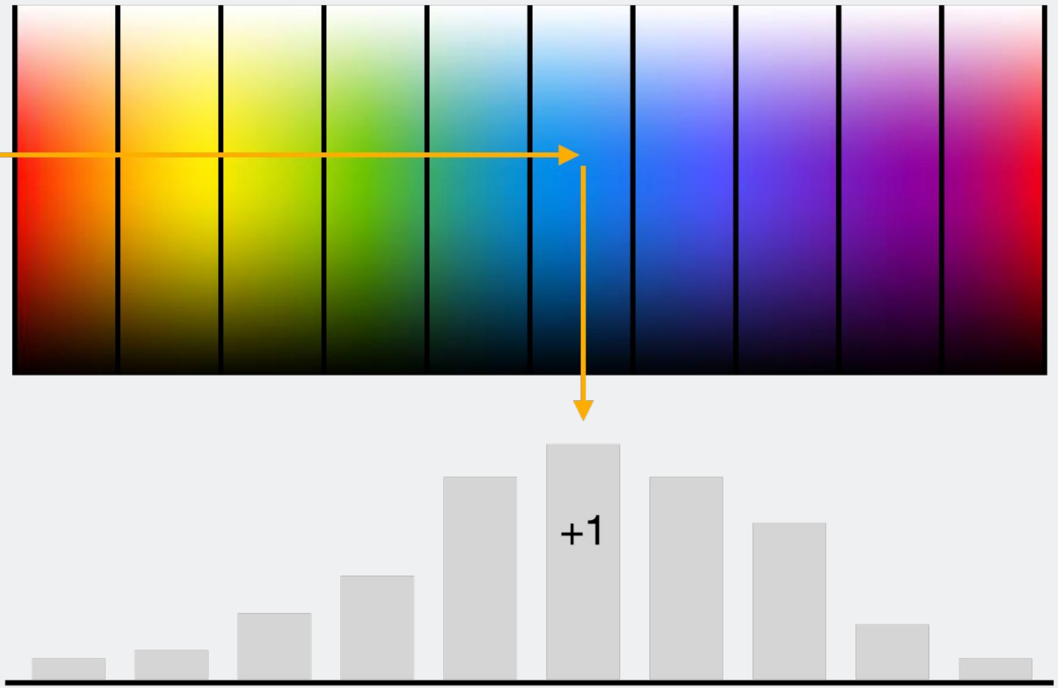
Image Feature: Color Histogram



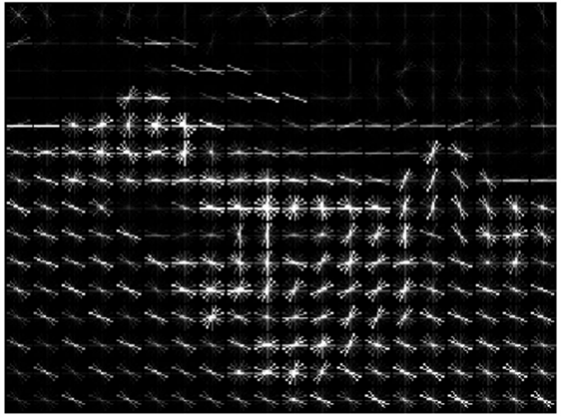
Ignores texture,
spatial positions

Image Feature: HoG (Histogram of Oriented Gradients)

1. Compute edge direction/ strength at each pixel
2. Divide image into 8x8 regions
3. Within each region compute a histogram of edge direction weighted by edge strength

Lowe, “Object recognition from local scale-invariant features,” ICCV 1999
Dalal and Triggs, “Histograms of oriented gradients for human detection,” CVPR 2005

Image Feature: HoG (Histogram of Oriented Gradients)



1. Compute edge direction/ strength at each pixel
2. Divide image into 8x8 regions
3. Within each region compute a histogram of edge direction weighted by edge strength

Example: 320x240 image gets divided into 40x30 bins;

9 directions per bin;

feature vector has 30*40*9 = 10,800 numbers

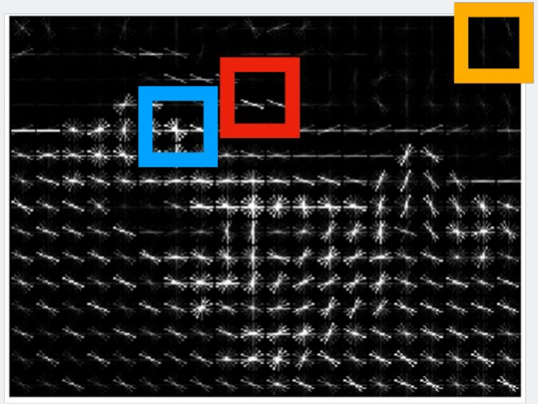
Lowe, "Object recognition from local scale-invariant features," ICCV 1999
Dalal and Triggs, "Histograms of oriented gradients for human detection," CVPR 2005

Image Feature: HoG (Histogram of Oriented Gradients)

Weak edges

Strong diagonal edges

Edges in all directions



1. Compute edge direction/ strength at each pixel
2. Divide image into 8x8 regions
3. Within each region compute a histogram of edge direction weighted by edge strength

Capture texture and position, robust to small image changes

Example: 320x240 image gets divided into 40x30 bins;

9 directions per bin;

feature vector has 30*40*9 = 10,800 numbers

Lowe, "Object recognition from local scale-invariant features," ICCV 1999
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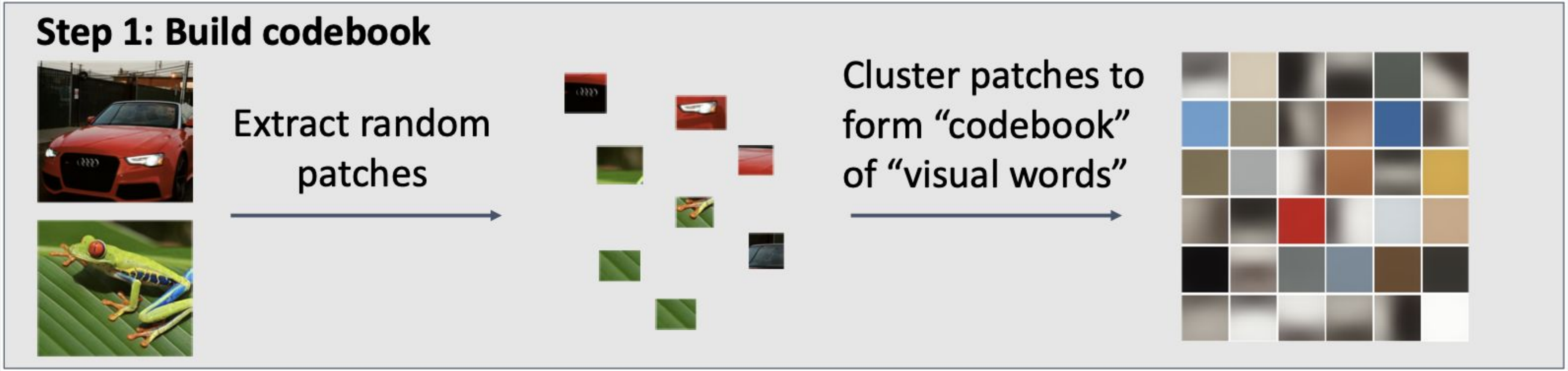
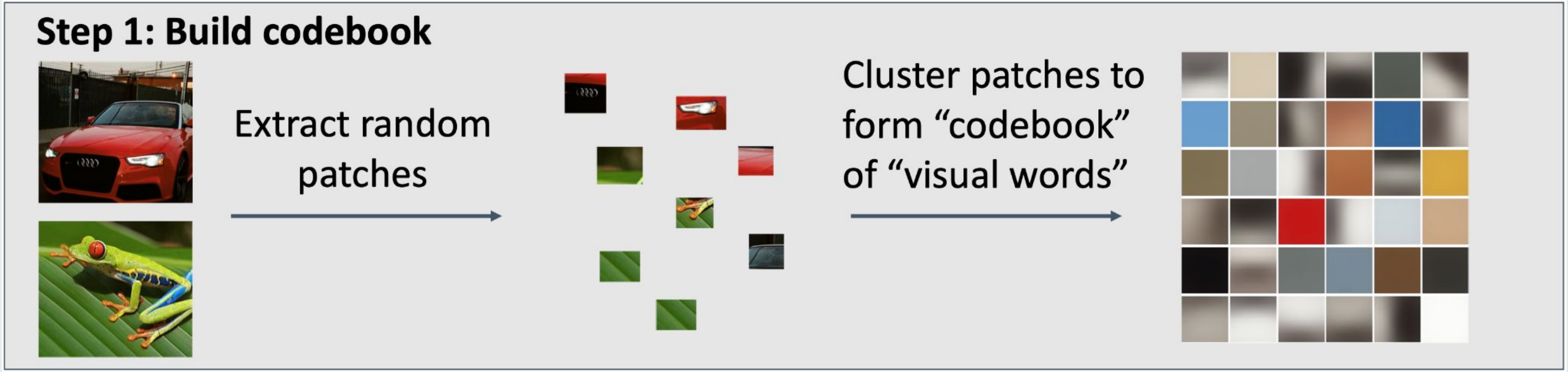
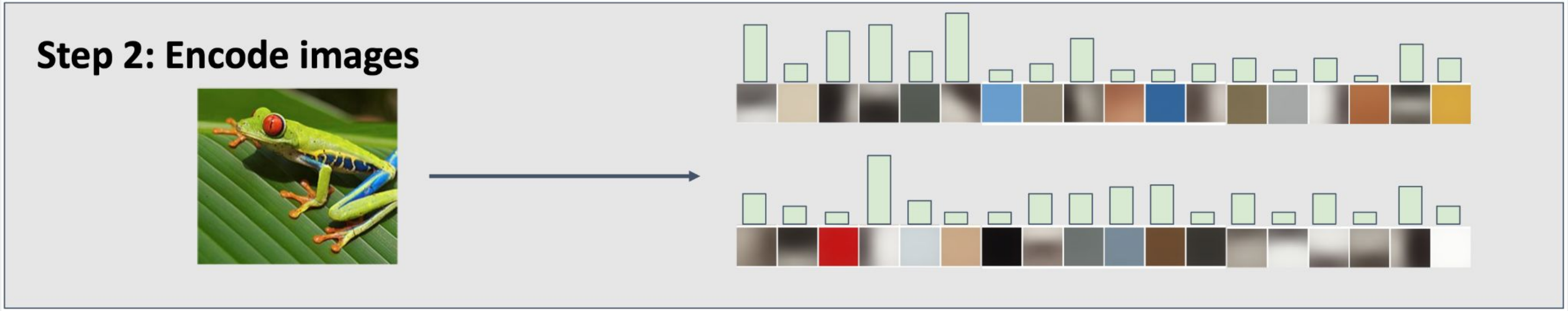
Image Feature: Bag of Words (Data Driven)



Fei-Fei and Perona, “A bayesian hierarchical model for learning natural scene categories,” CVPR 2005

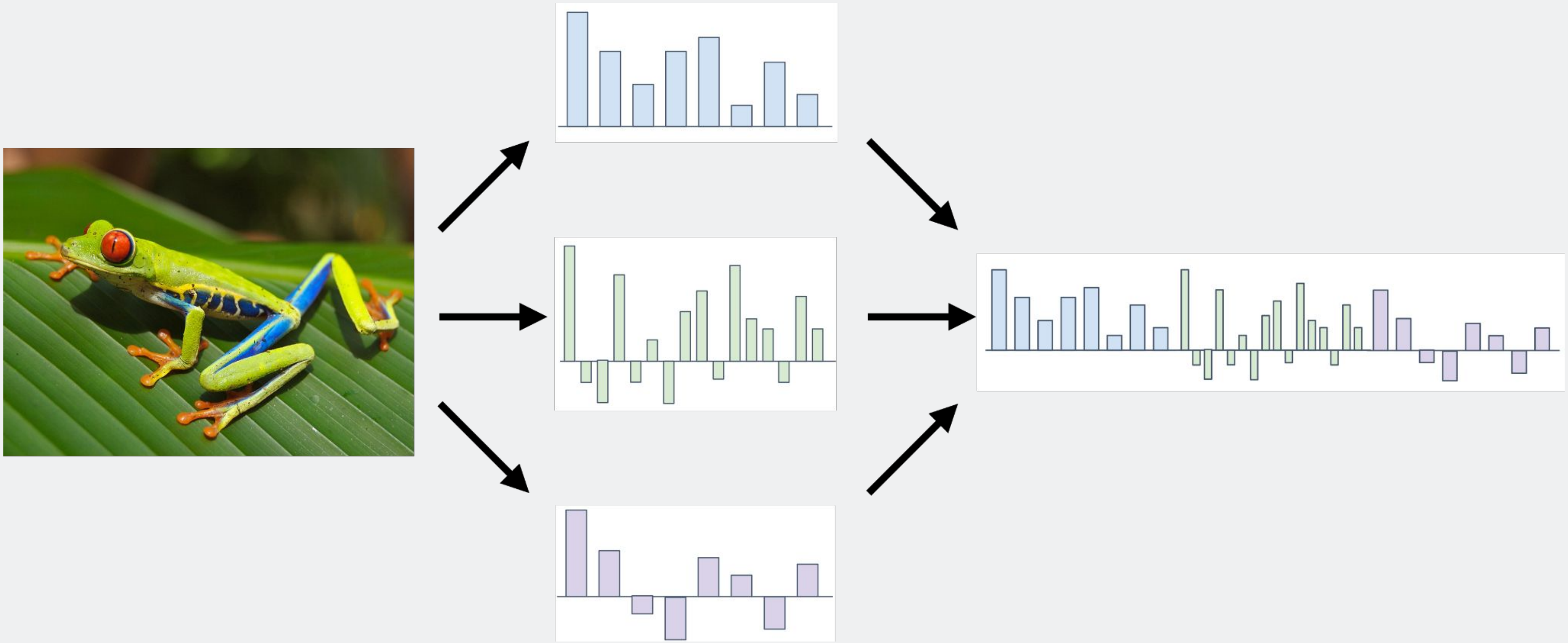
Image Feature: Bag of Words (Data Driven)





Fei-Fei and Perona, "A bayesian hierarchical model for learning natural scene categories," CVPR 2005

Image Features



Example: Winner of 2011 ImageNet Challenge

Low-level feature extraction ≈ 10k patches per image

- SIFT: 128-dims
- Color: 96-dim

Reduced to 64-dim with PCA

FV extraction and compression:

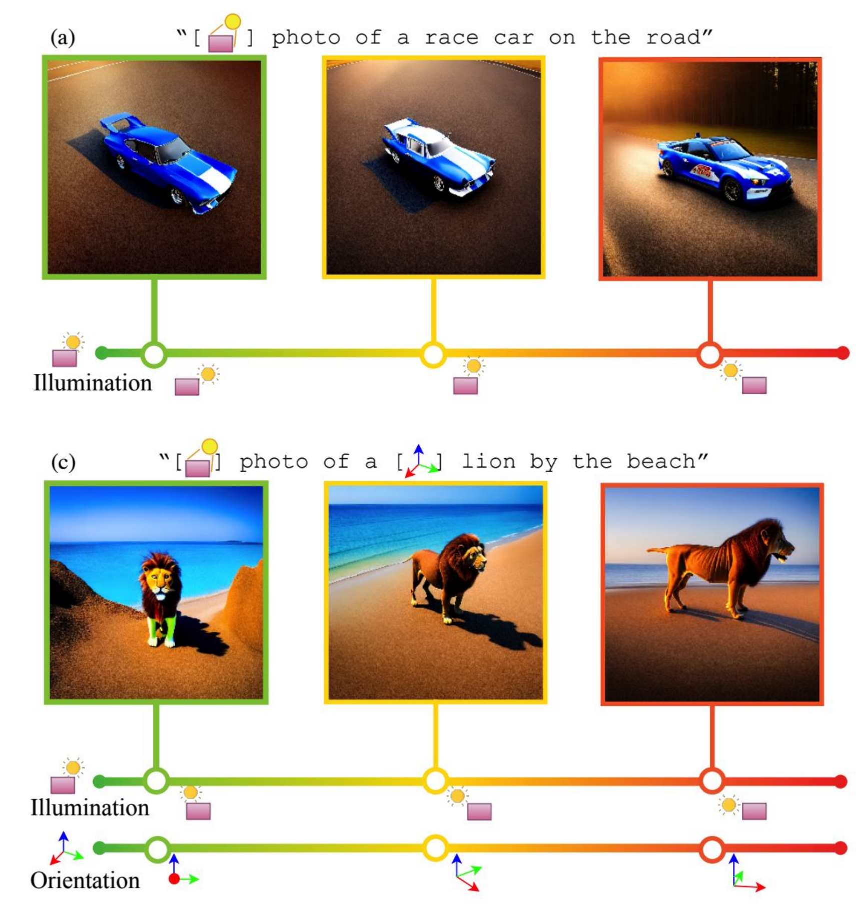
- N=1024 Gaussians, R=4 regions → 520K dim x 2
- Compression: G=8, b=1 bit per dimension

One-vs-all SVM learning with SGD

Late fusion of SIFT and color systems

F. Perronnin, J. Sánchez, “Compressed Fisher vectors for LSVRC”, PASCAL VOC / ImageNet workshop, ICCV, 2011.

Example: 2024 CVPR accepted paper



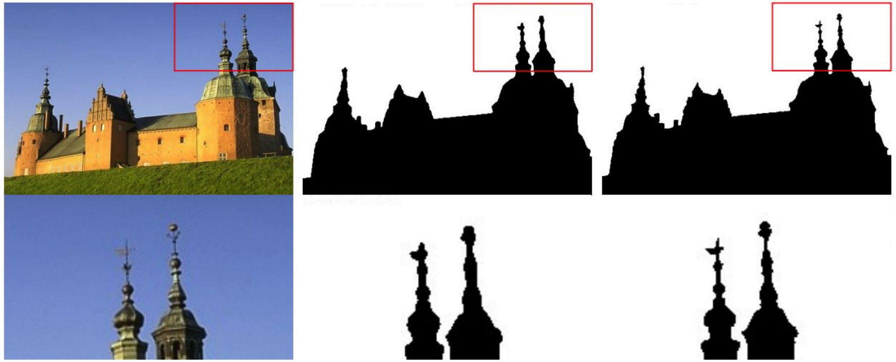
"Text-to-image" 3D Words

Cheng, T. Y., Gadelha, M., Groueix, T., Fisher, M., Mech, R., Markham, A., & Trigoni, N. (2024). Learning Continuous 3D Words for Text-to-Image Generation. In Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition (pp. 6753-6762).

Example: 2024 CVPR accepted paper

(a) (b) (c) (d)

Figure 1. Visual comparison of 500 superpixels resulting from (a, c) ETPS [previous], (b, d) HHTS [proposed] segmentation.



(a) (b) (c)

Figure 2. Visual comparison of semantic segment masks (a) original image, (b) semantic segment (SAM ViT-H) [previous] and (c) refined semantic segment (SAM + HHTS) [proposed]

- Hierarchical Histogram Threshold Segmentation
- “Fine-tuning” segmentation masks

Chang, T. V., Seibt, S., & von Rymon Lipinski, B. (2024). Hierarchical Histogram Threshold Segmentation-Auto-terminating High-detail Oversegmentation. In Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition (pp. 3195-3204).

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Image Features

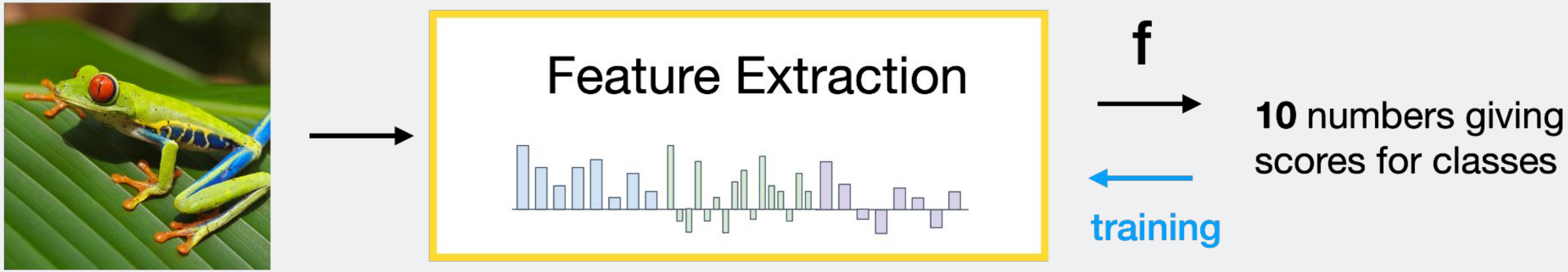
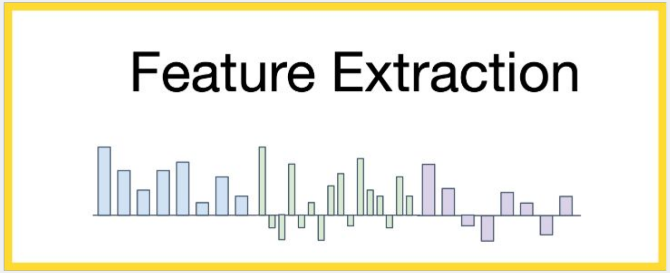


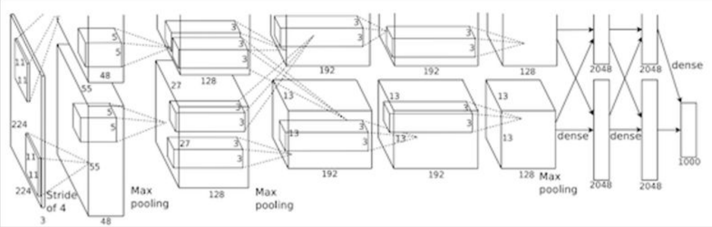
Image Features vs. Neural networks



f

10 numbers giving scores for classes

training



Krizhevsky, Sutskever, and Hinton, "Imagenet classification with deep convolutional neural networks", NIPS 2012.

10 numbers giving scores for classes

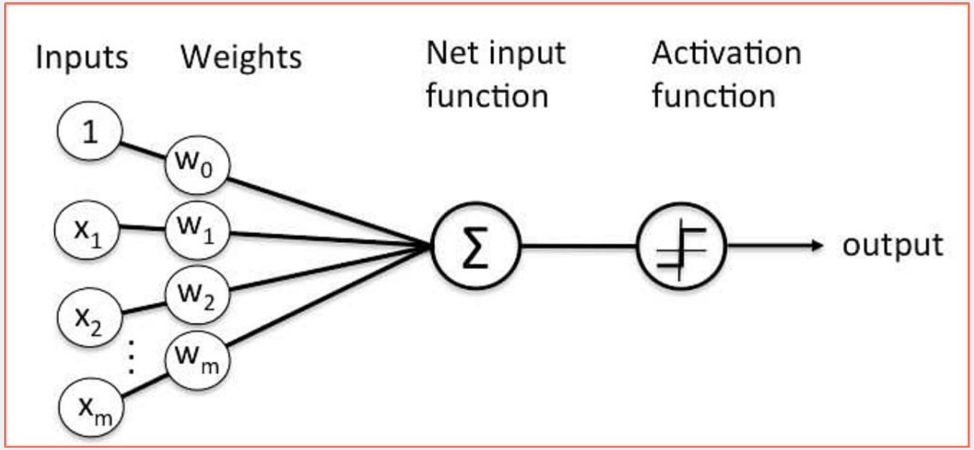
training

Neural Networks (Overview)

Input: $x \in \mathbb{R}^D$

Output: $f(x) \in \mathbb{R}^C$

Rosenblatt's Perceptron



•A set of *synapses* each of which is characterized by a *weight* (which includes a *bias*).

•An *adder*

•An *activation function* (e.g., Rectified Linear Unit/ReLU, Sigmoid function, etc.)

$$y_k = \phi\left(\sum_{j=1}^{m} w_{kj} x_j + b_k\right)$$

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Neural Networks

Input: $x \in \mathbb{R}^D$ **Output:** $f(x) \in \mathbb{R}^C$

Before: Linear Classifier: $f(x) = Wx + b$

Learnable parameters: $W \in \mathbb{R}^{D \times C}, b \in \mathbb{R}^C$

Neural Networks

Input: $x \in \mathbb{R}^D$ **Output:** $f(x) \in \mathbb{R}^C$

Before: Linear Classifier: $f(x) = Wx + b$
Learnable parameters: $W \in \mathbb{R}^{D \times C}, b \in \mathbb{R}^C$

Feature Extraction

Linear Classifier

Now: Two-Layer Neural Network: $f(x) = W_2 \max(0, W_1 x + b_1) + b_2$
Learnable parameters: $W_1 \in \mathbb{R}^{H \times D}, b_1 \in \mathbb{R}^H, W_2 \in \mathbb{R}^{C \times H}, b_2 \in \mathbb{R}^C$

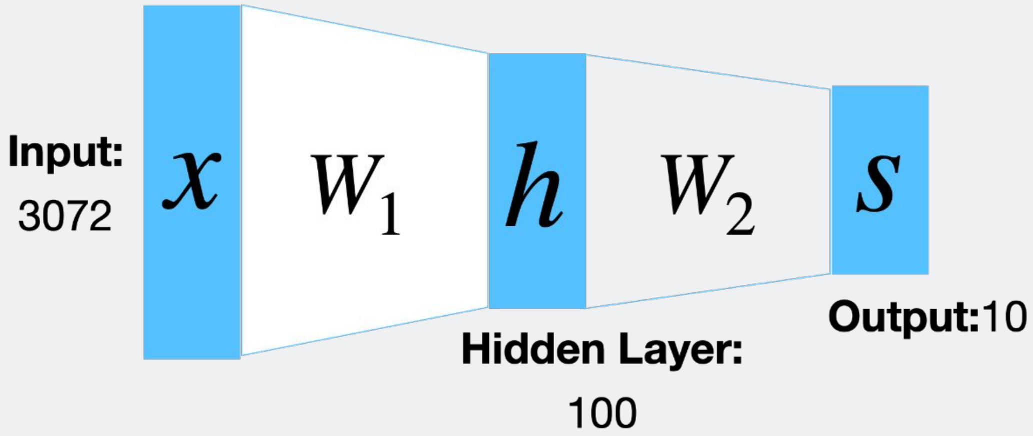
Or Three-Layer Neural Network:

$$f(x) = W_3 \max(0, W_2 \max(0, W_1 x + b_1) + b_2) + b_3$$

Neural Networks

Before: Linear Classifier: $f(x) = Wx + b$

Now: Two-Layer Neural Network: $f(x) = W_2 \max(0, W_1 x + b_1) + b_2$



$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

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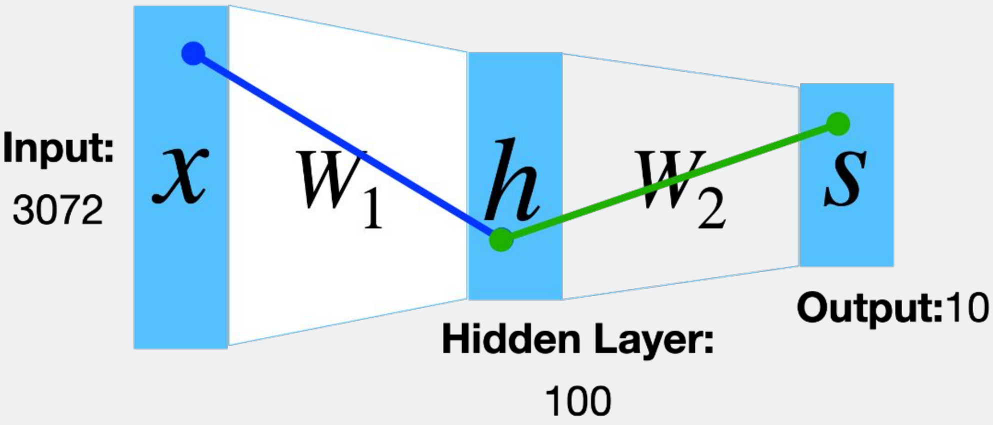
Neural Networks

Before: Linear Classifier: $f(x) = Wx + b$

Now: Two-Layer Neural Network: $f(x) = W_2 \max(0, W_1 x + b_1) + b_2$

Element (i,j) of W_1 gives the effect on h_i from x_j



Element (i,j) of W_2 gives the effect on s_i from h_j

$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

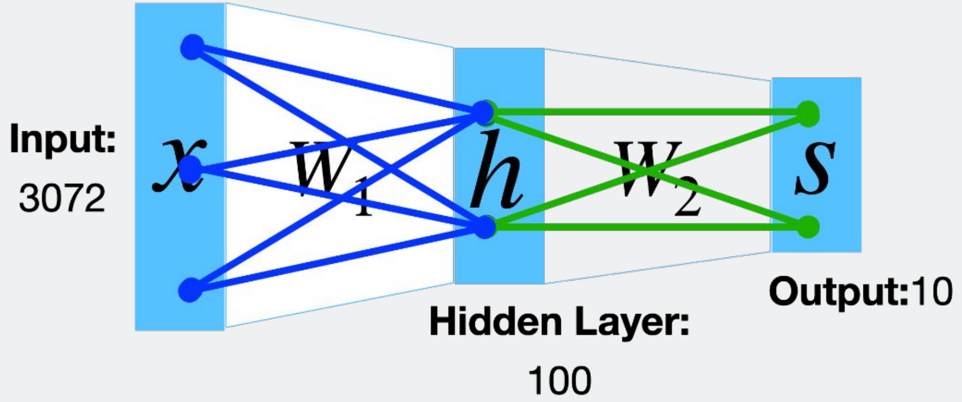
Neural Networks

Before: Linear Classifier: $f(x) = Wx + b$

Now: Two-Layer Neural Network: $f(x) = W_2 \max(0, W_1 x + b_1) + b_2$

Element (i, j) of W_1 gives the effect on h_i from x_j

All elements of x affect all elements of h



Element (i, j) of W_2 gives the effect on s_i from h_j

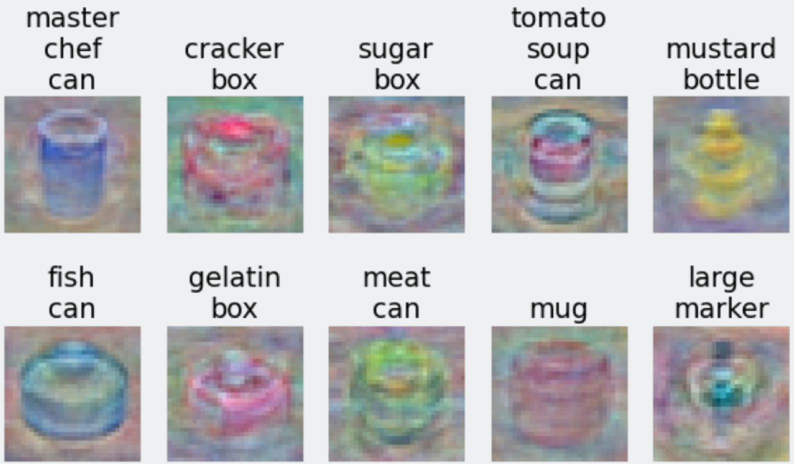
All elements of h affect all elements of s

Fully-connected neural network also "Multi-Layer Perceptron" (MLP)

Neural Networks

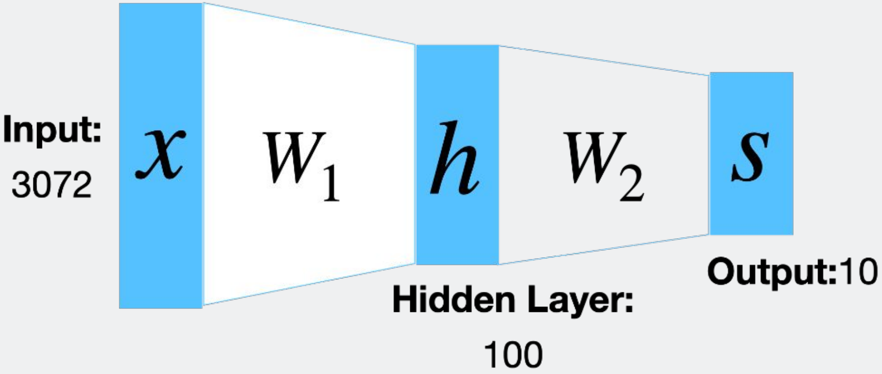
Recall:

Linear classifier: One template per class



Before: Linear score function

Now: Two-Layer Neural Network:



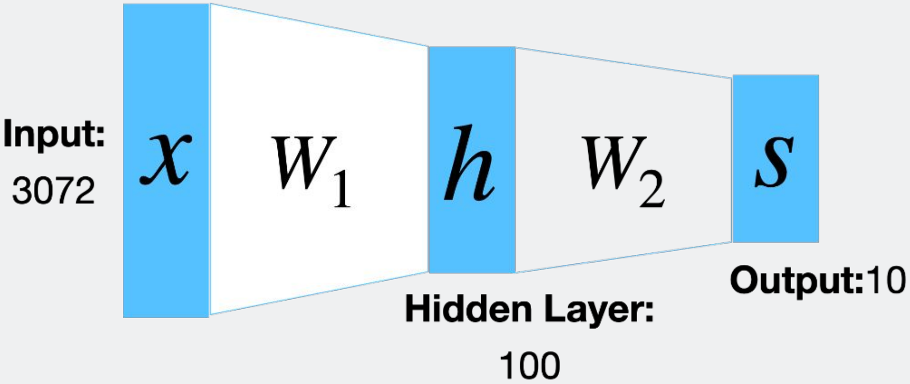
$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H\times D}, W_2 \in \mathbb{R}^{C\times H}$$

Neural Networks

Neural net: first layer is bank of templates;
Second layer recombines templates

Before: Linear score function

Now: Two-Layer Neural Network:



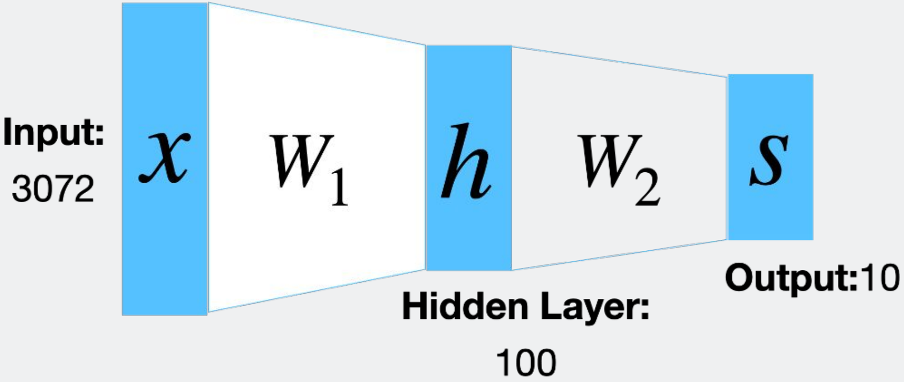
$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Neural Networks

Can use different templates to cover multiple modes of a class!

Before: Linear score function

Now: Two-Layer Neural Network:



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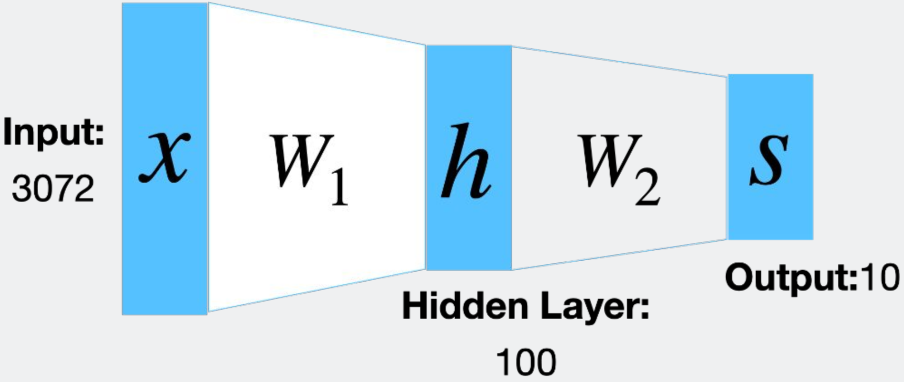
Before: Linear score function

Now: Two-Layer Neural Network:



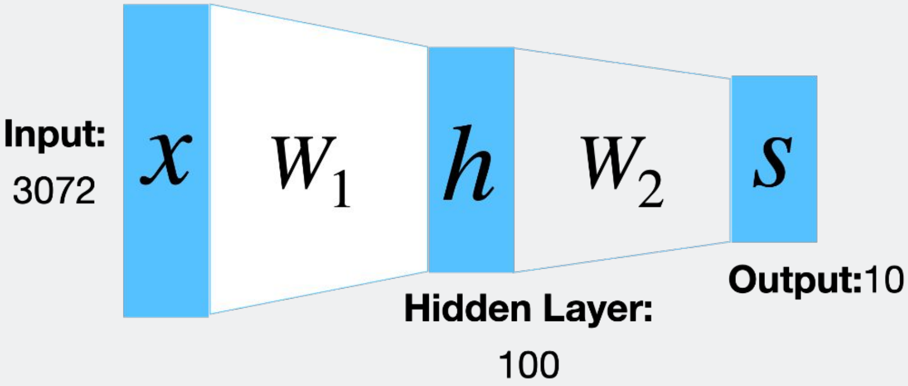
$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Neural Networks

"Distributed representation": Most templates not interpretable!

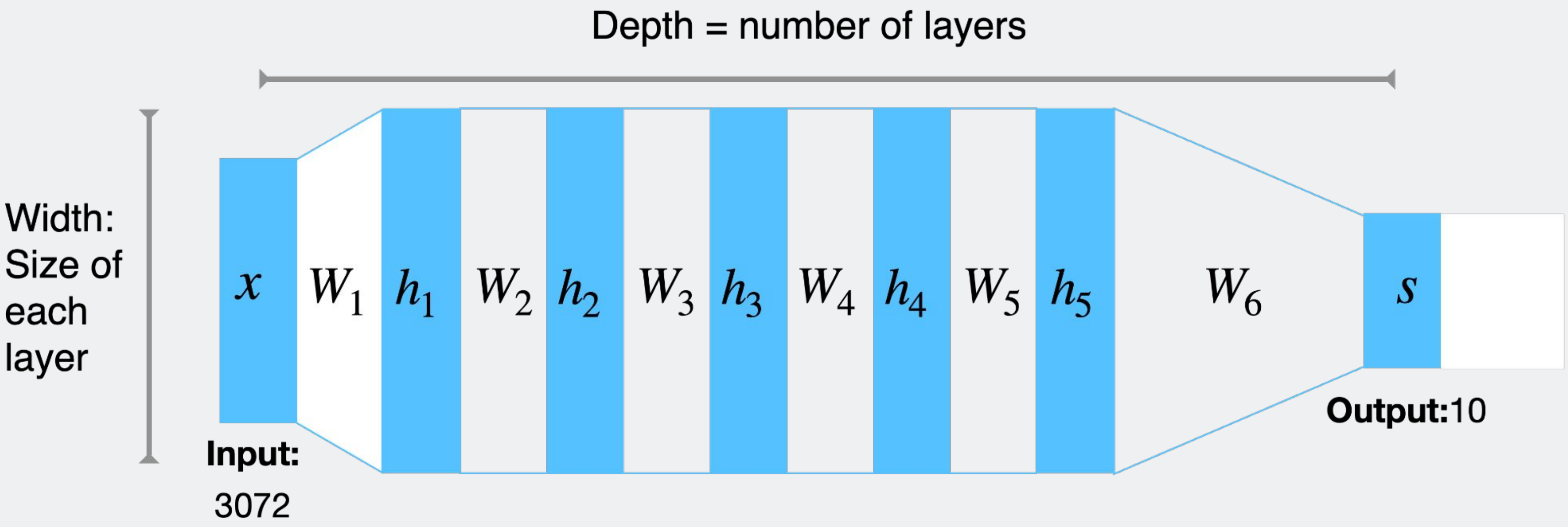
Before: Linear score function

Now: Two-Layer Neural Network:



$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H\times D}, W_2 \in \mathbb{R}^{C\times H}$$

Deep Neural Networks



$$s = W_6 \max(0, W_5 \max(0, W_4 \max(0, W_3 \max(0, W_2 \max(0, W_1 x)))))$$

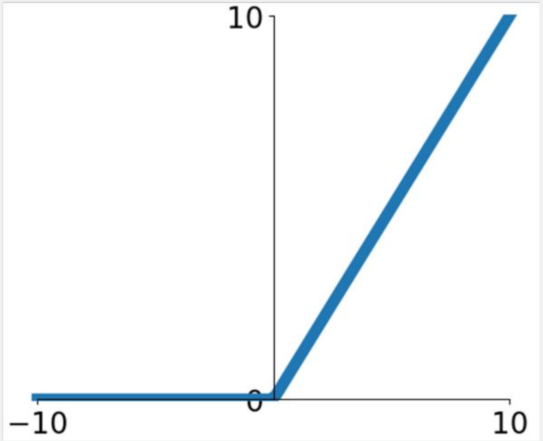
Neural Networks: Activation Functions

2-Layer Neural Network

$$f(x) = W_2 \max(0, W_1 x + b_1) + b_2$$

The auction $ReLU(z) = \max(0, z)$ is called "Rectified Linear Unit"

This is called the **activation function** of the neural network



Q: What happens if we build a neural network with <u>no</u> activation function?

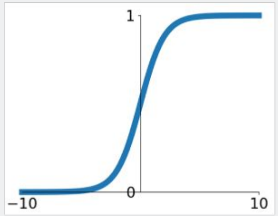
Aha Slides
(In-class participation)

https://ahaslides.com/0Z9LZ

Activation Functions

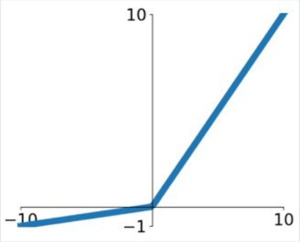
Sigmoid

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



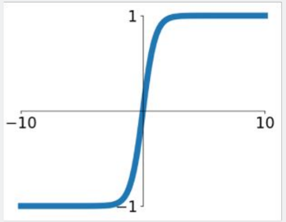
Leaky ReLU

$$\max(0.2x, x)$$



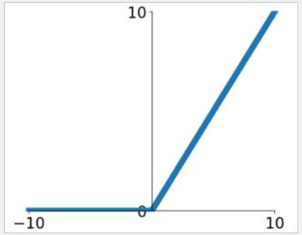
tanh

$$\tanh(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$



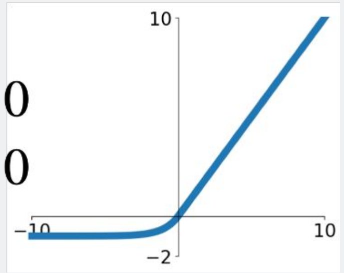
ReLU

$$\max(0,x)$$



ELU

$$f(x) = \begin{cases} x, & x > 0 \\ \alpha(\exp(x) -), & x \leq 0 \end{cases}$$



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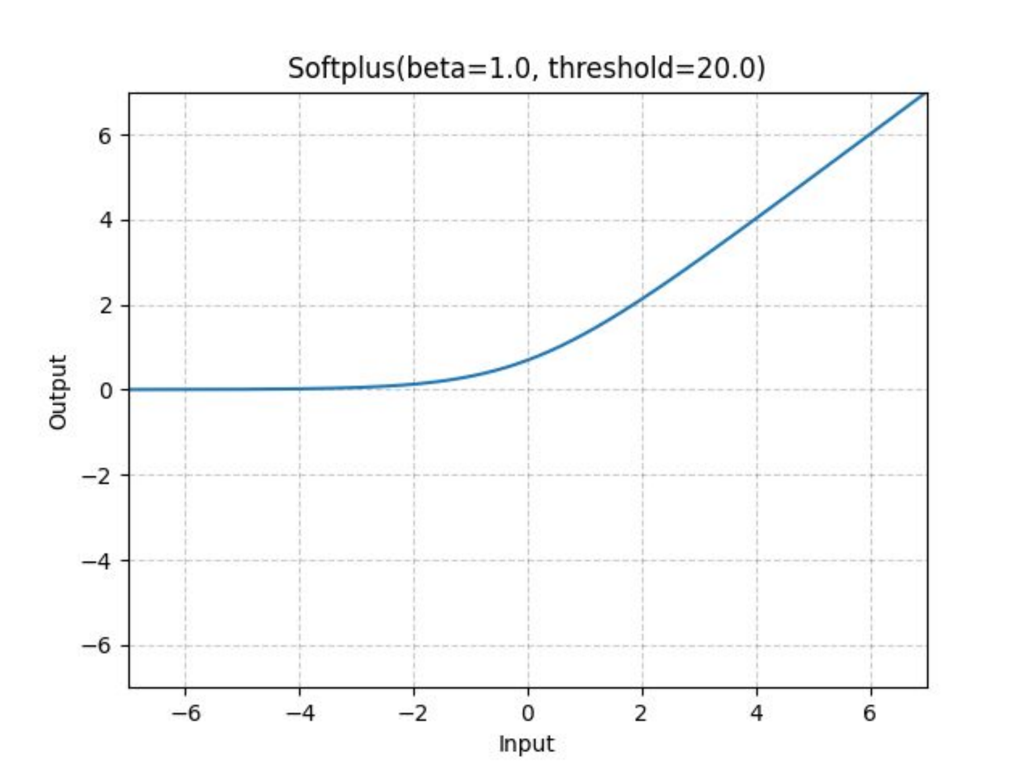
Activation Functions

Softplus

$$\log(1 + \exp(x))$$

https://pytorch.org/docs/stable/generated/torch.nn.Softplus.html



Activation Functions

Maxout

$$\max(w_1^T x + b_1, w_2^T x + b_2)$$

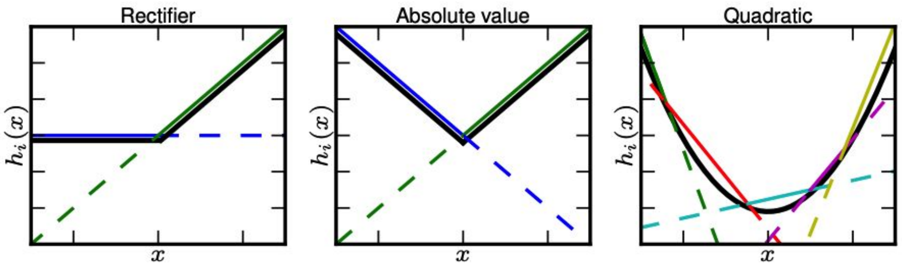


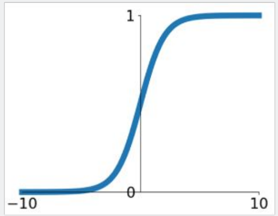
Figure 1. Graphical depiction of how the maxout activation function can implement the rectified linear, absolute value rectifier, and approximate the quadratic activation function. This diagram is 2D and only shows how maxout behaves with a 1D input, but in multiple dimensions a maxout unit can approximate arbitrary convex functions.

https://proceedings.mlr.press/v28/goodfellow13.pdf

Activation Functions

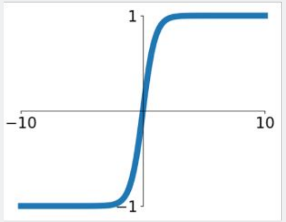
Sigmoid

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



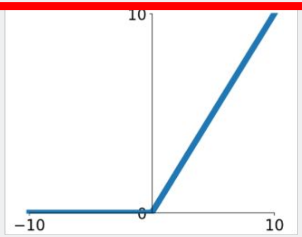
tanh

$$\tanh(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$



ReLU

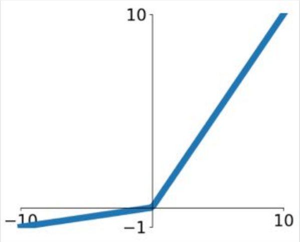
$\max(0,x)$



ReLU is a good default choice for most problems

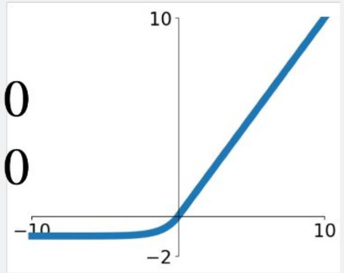
Leaky ReLU

$\max(0.2x, x)$



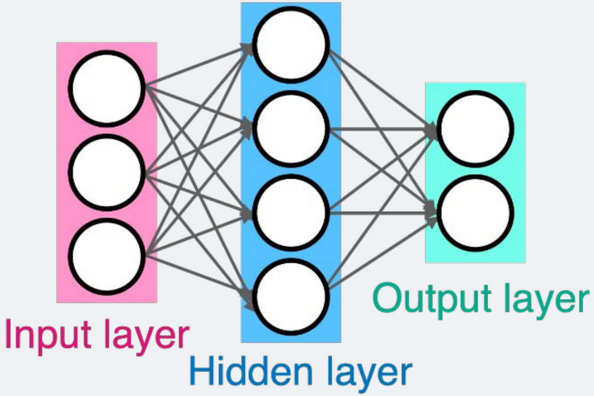
ELU

$$f(x) = \begin{cases} x, & x > 0 \\ \alpha(\exp(x) -), & x \leq 0 \end{cases}$$



ROBOTICS

Neural Network in 20 Lines



Initialize weights and data

Compute loss (Sigmoid activation, L2 loss)

Compute gradients

SGD step

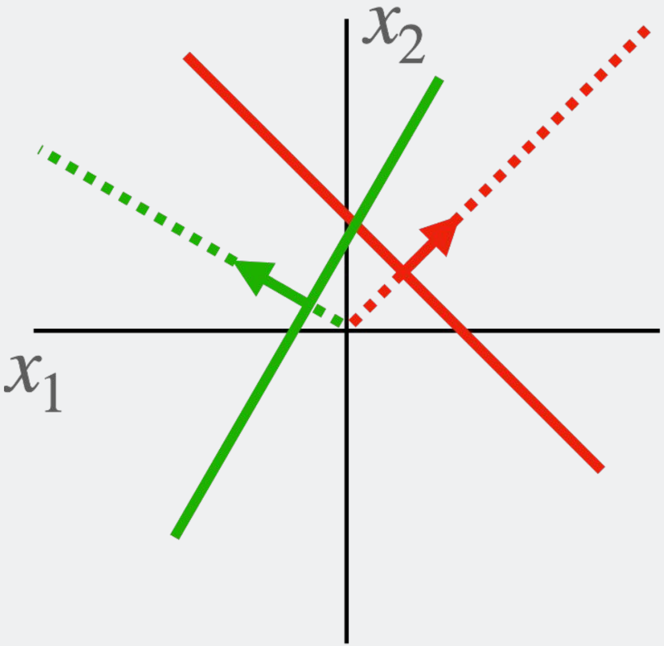
```python
import numpy as np
from numpy.random import randn

N, Din, H, Dout = 64, 1000, 100, 10
x, y = randn(N, Din), randn(N, Dout)
w1, w2 = randn(Din, H), randn(H, Dout)
for t in range(10000):
  h = 1.0 / (1.0 + np.exp(-x.dot(w1)))
  y_pred = h.dot(w2)
  loss = np.square(y_pred - y).sum()
  dy_pred = 2.0 * (y_pred - y)
  dw2 = h.T.dot(dy_pred)
  dh = dy_pred.dot(w2.T)
  dw1 = x.T.dot(dh * h * (1 - h))
  w1 -= 1e-4 * dw1
  w2 -= 1e-4 * dw2
```

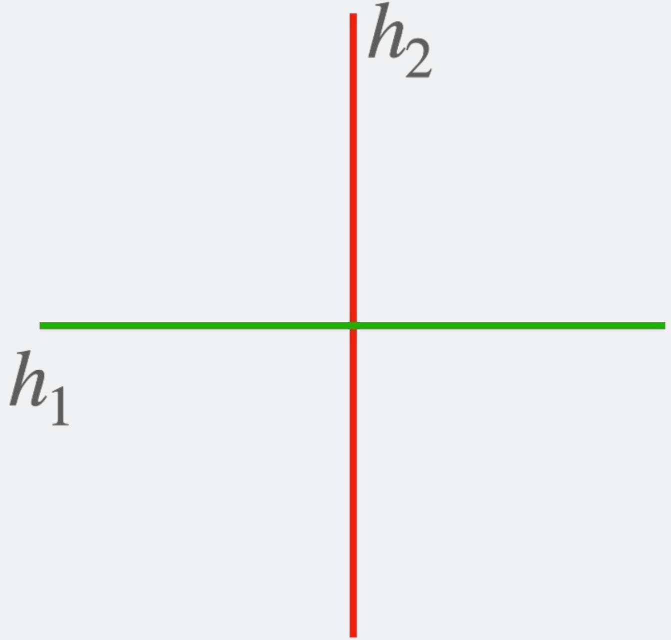
Space Warping

Space Warping

Consider a linear transform: $h = Wx + b$
where x, b, h are each 2-dimensional

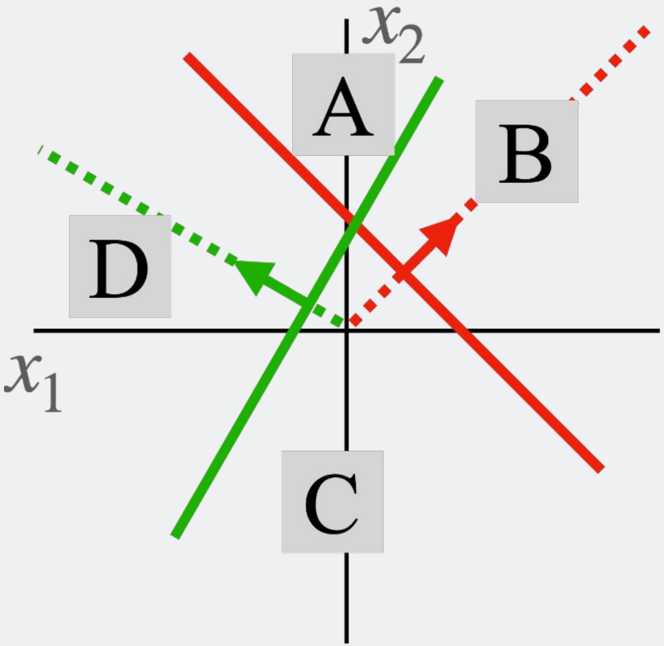


Feature transform:
$h = Wx + b$

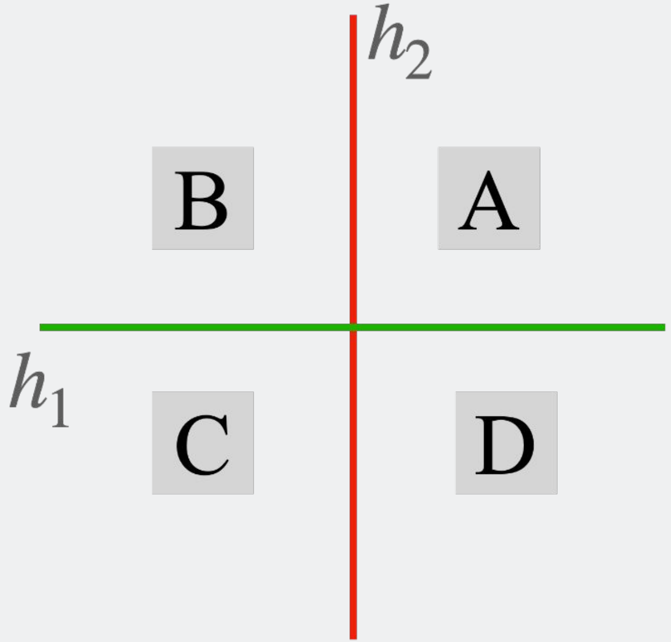


Space Warping

Consider a linear transform: $h = Wx + b$
where x, b, h are each 2-dimensional

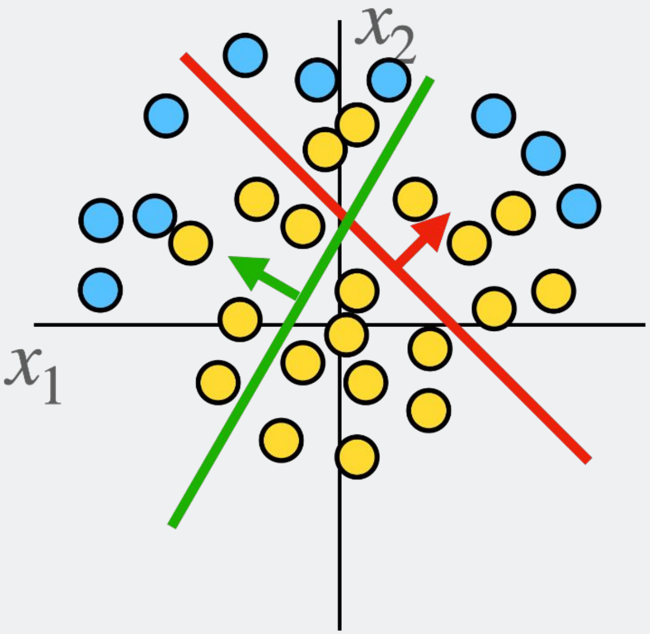


Feature transform:
$h = Wx + b$



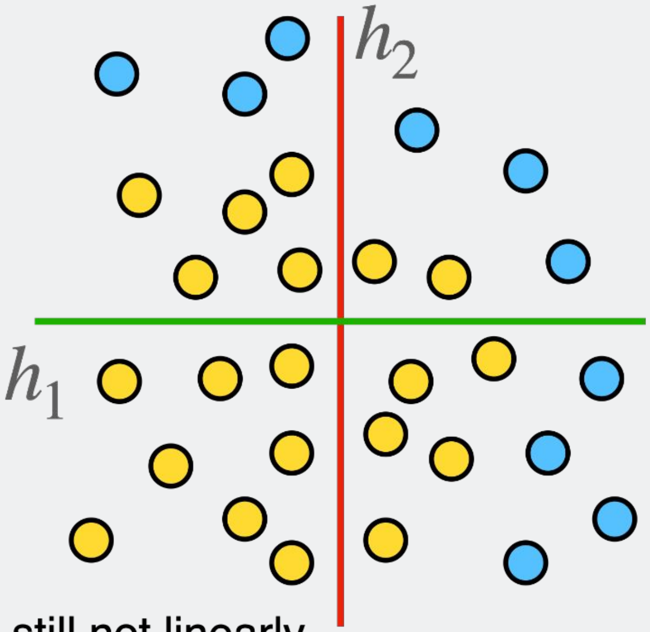
Space Warping

Points not linearly separable
in original space



Feature transform:
$h = Wx + b$

Consider a linear transform: $h = Wx + b$
where x, b, h are each 2-dimensional

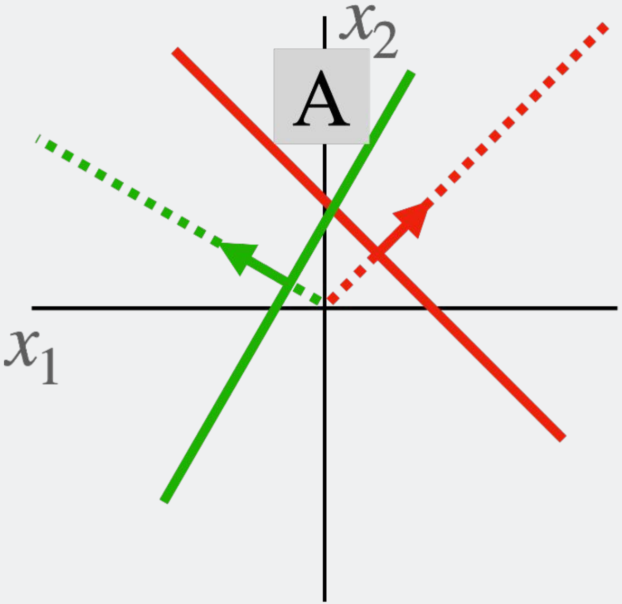


Points still not linearly
separable in feature space

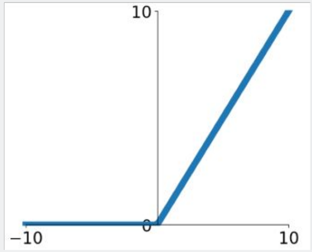
ROBOTICS

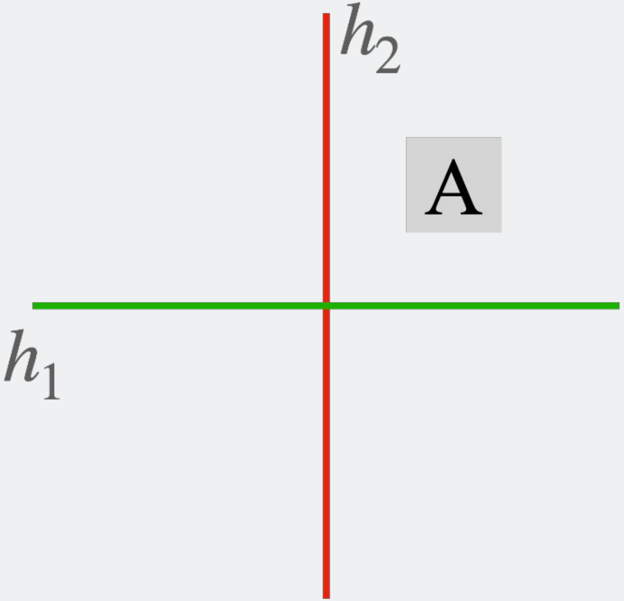
Space Warping

Consider a neural net hidden layer: $h = ReLU(Wx + b) = \max(0, Wx + b)$ where x, b, h are each 2-dimensional



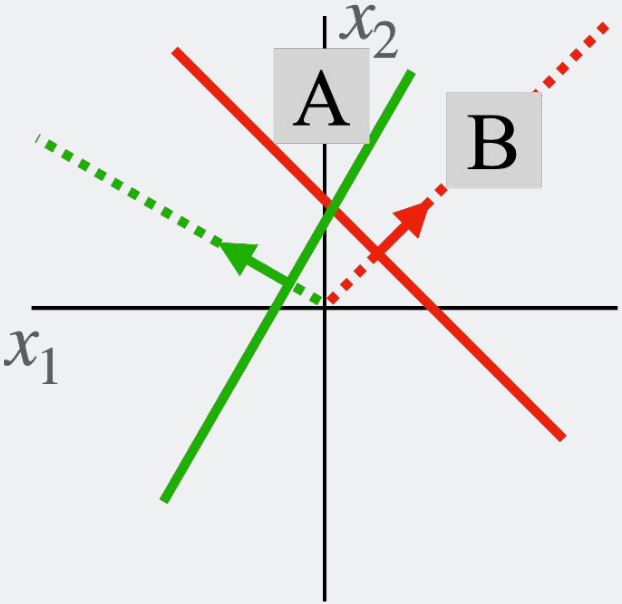
Feature transform:
$h = ReLU(Wx + b)$





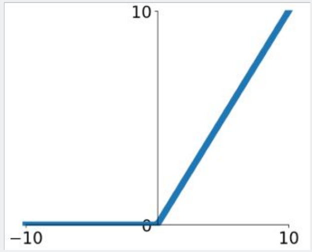
Space Warping

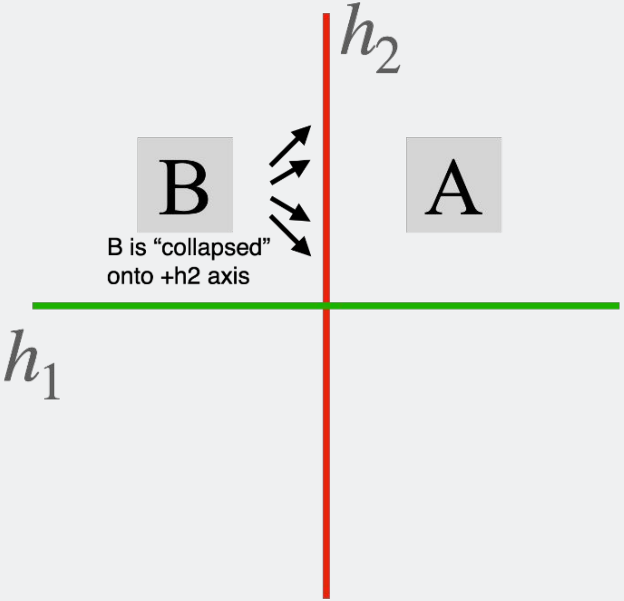
Consider a neural net hidden layer: $h = ReLU(Wx + b)$ $= \max(0, Wx + b)$ where x, b, h are each 2-dimensional



Feature transform:

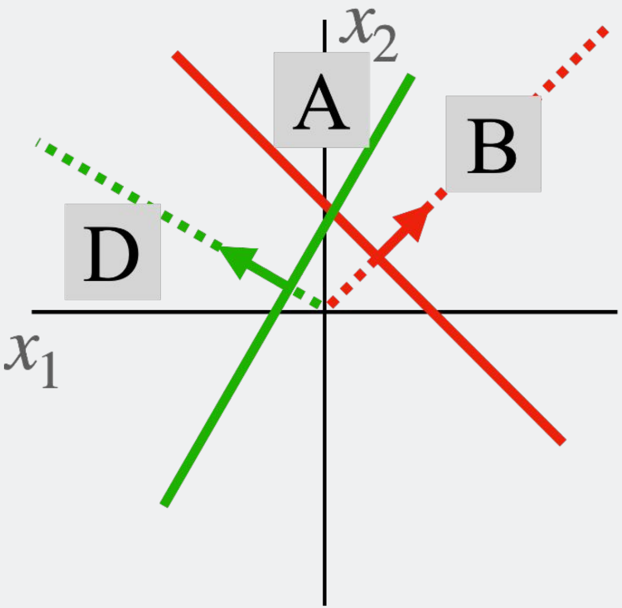
$h = ReLU(Wx + b)$



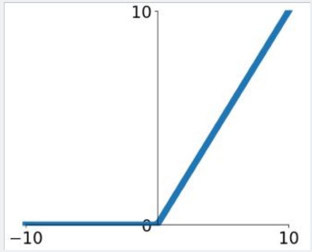


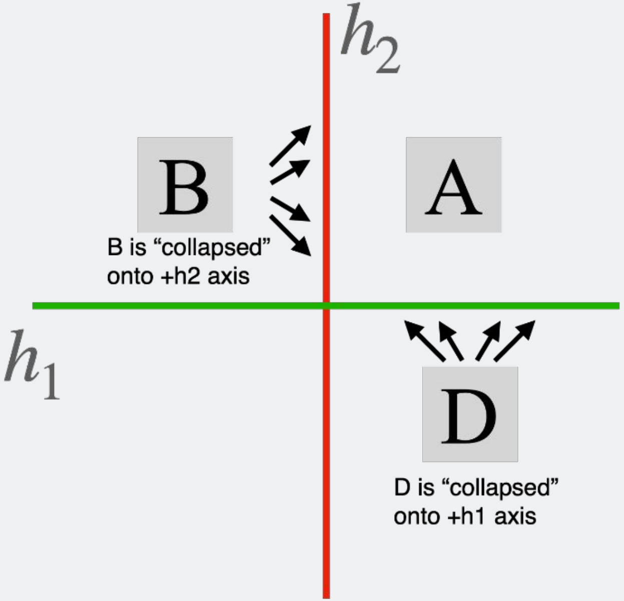
Space Warping

Consider a neural net hidden layer: $h = ReLU(Wx + b)$
$= \max(0, Wx + b)$ where x, b, h are each 2-dimensional



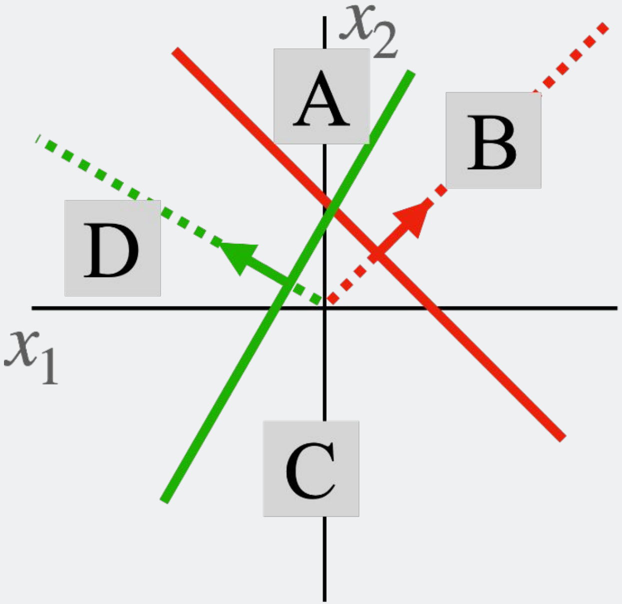
Feature transform:
$h = ReLU(Wx + b)$



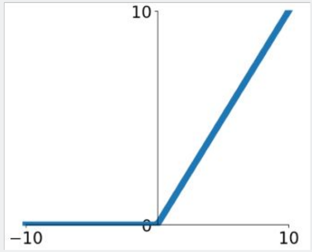


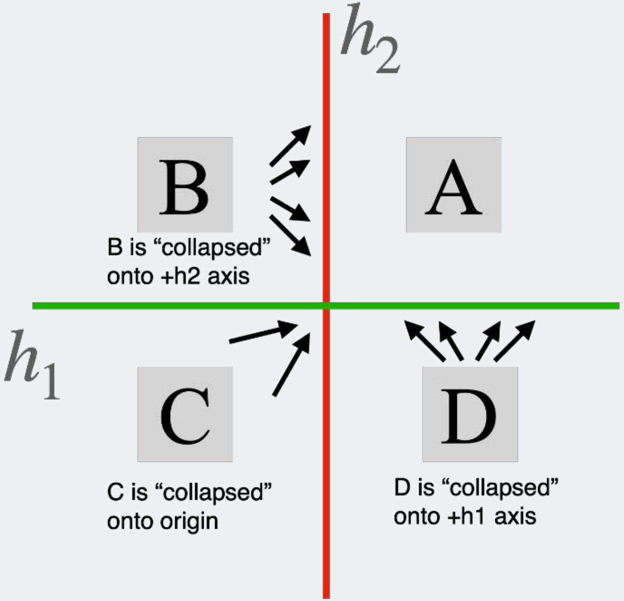
Space Warping

Consider a neural net hidden layer: $h = ReLU(Wx + b) = \max(0, Wx + b)$ where x, b, h are each 2-dimensional



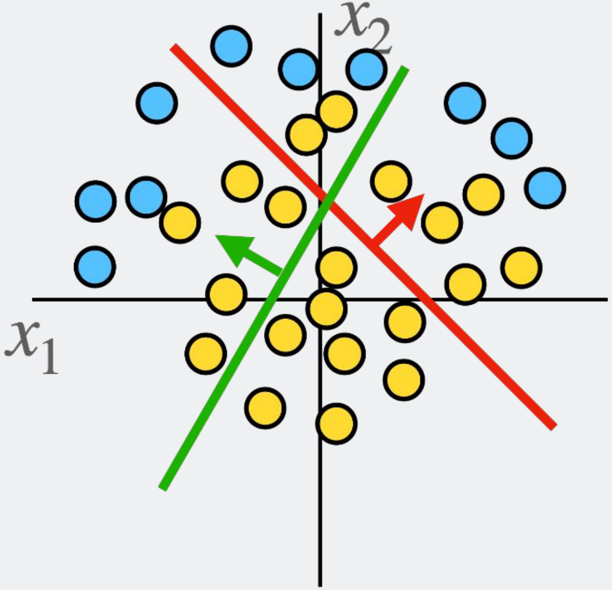
Feature transform:
$h = ReLU(Wx + b)$





Space Warping

Points not linearly separable in original space



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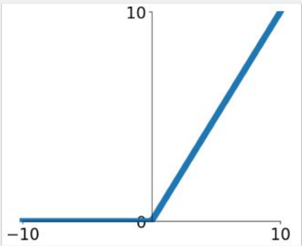
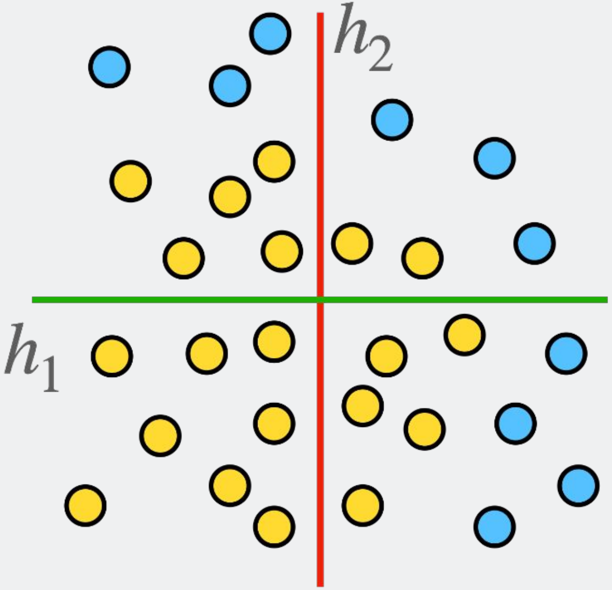
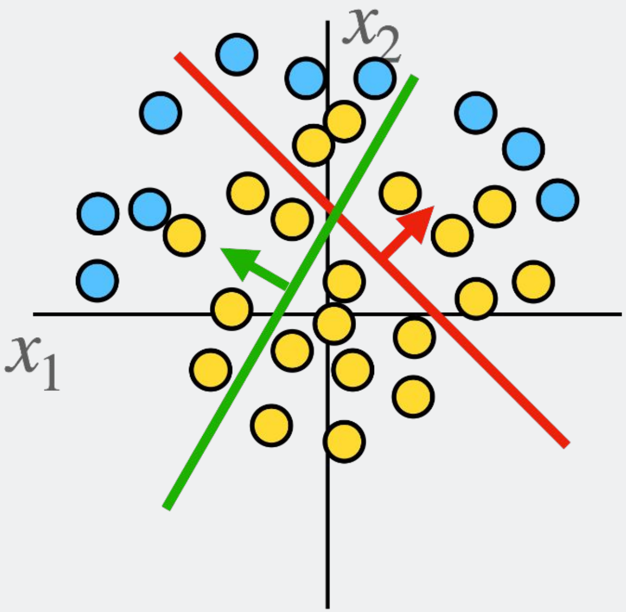
Feature transform:
$h = ReLU(Wx + b)$





Space Warping

Points not linearly separable in original space



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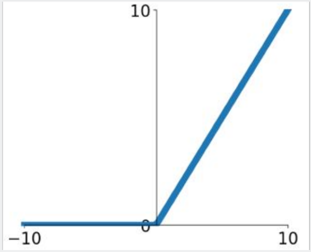
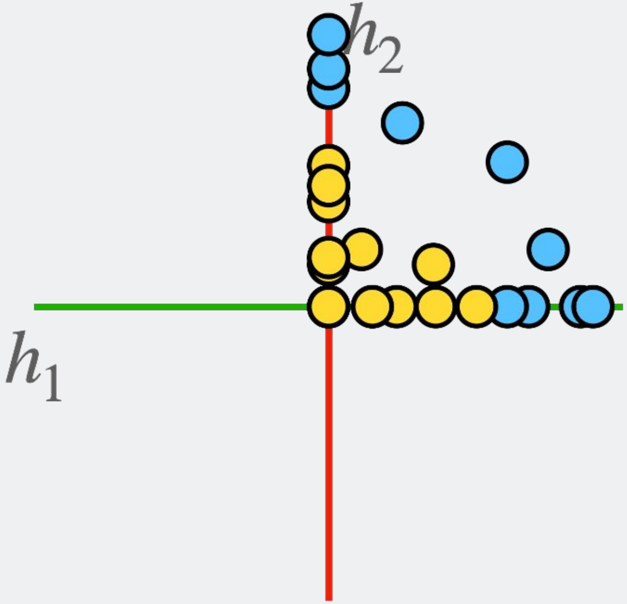
Feature transform:

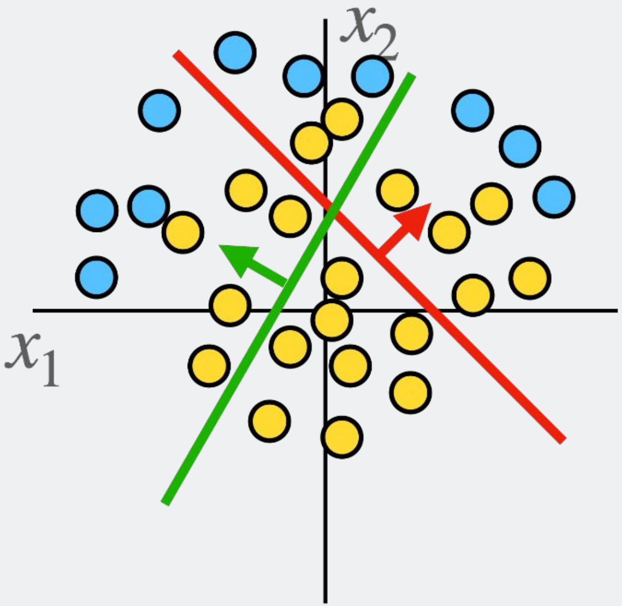
$h = ReLU(Wx + b)$





Space Warping

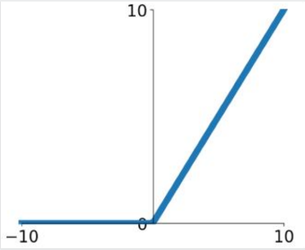
Points not linearly separable in original space

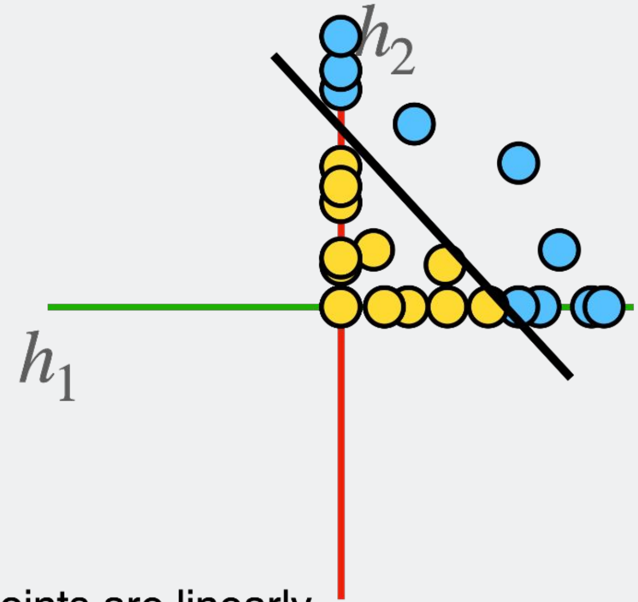


Consider a neural net hidden layer: $h = ReLU(Wx + b)$ $= \max(0, Wx + b)$ where x, b, h are each 2-dimensional

Feature transform:

$h = ReLU(Wx + b)$



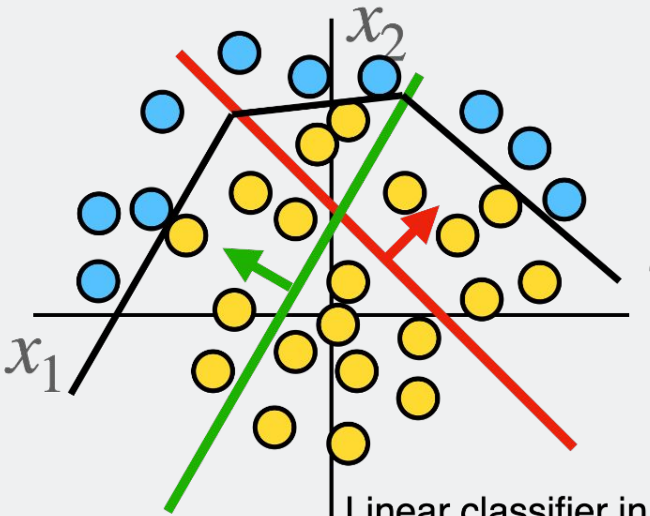


Points are linearly separable in feature space!

Space Warping

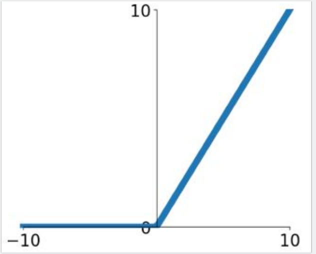
Points not linearly separable in original space

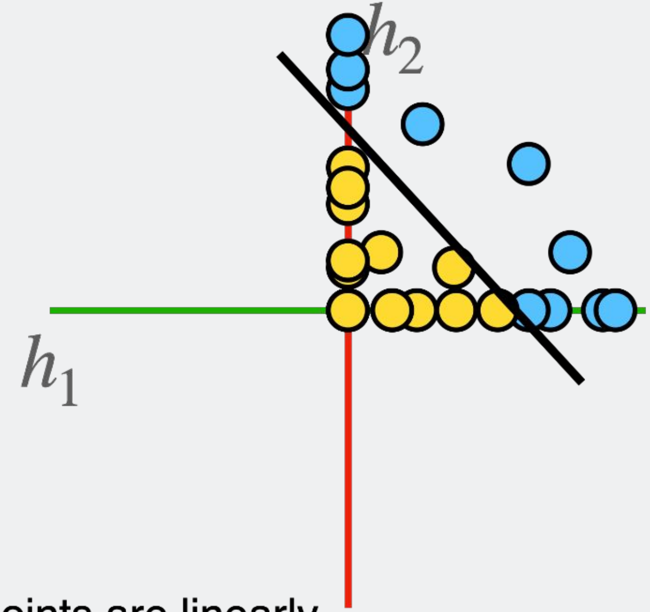
Consider a neural net hidden layer: $h = ReLU(Wx + b)$ $= \max(0, Wx + b)$ where x, b, h are each 2-dimensional



Linear classifier in feature space gives nonlinear classifier in original space

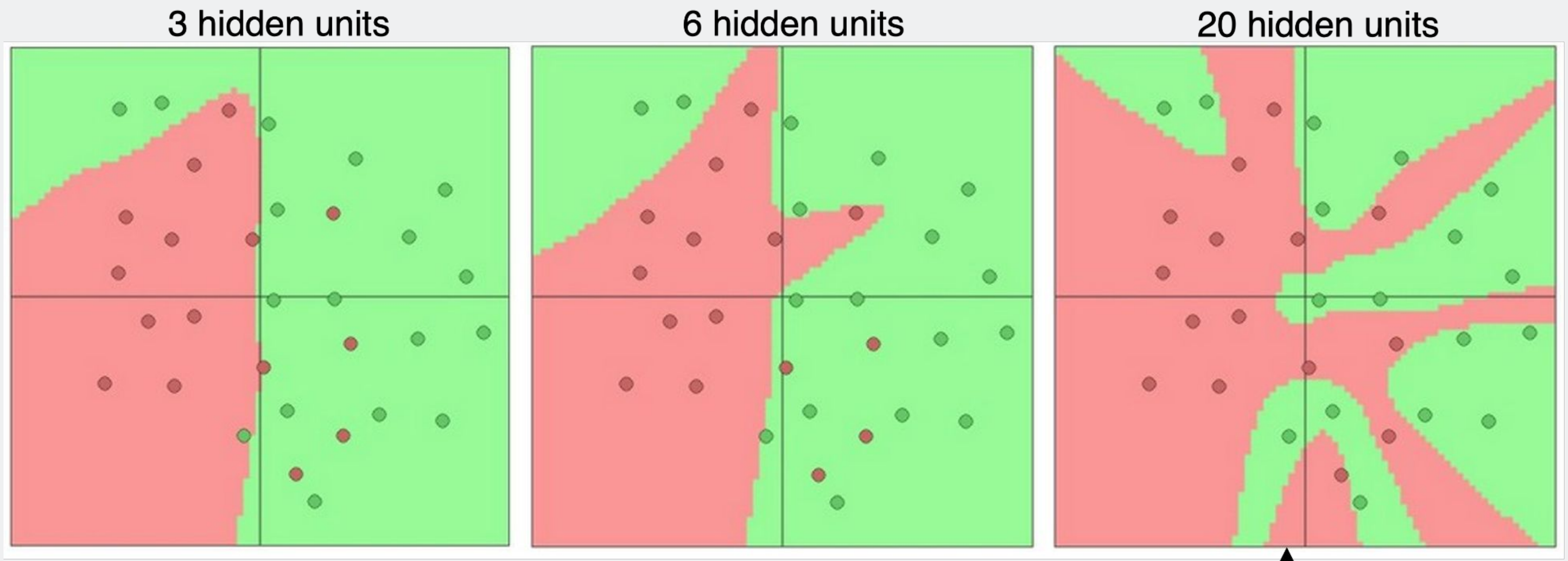
Feature transform:
$h = ReLU(Wx + b)$





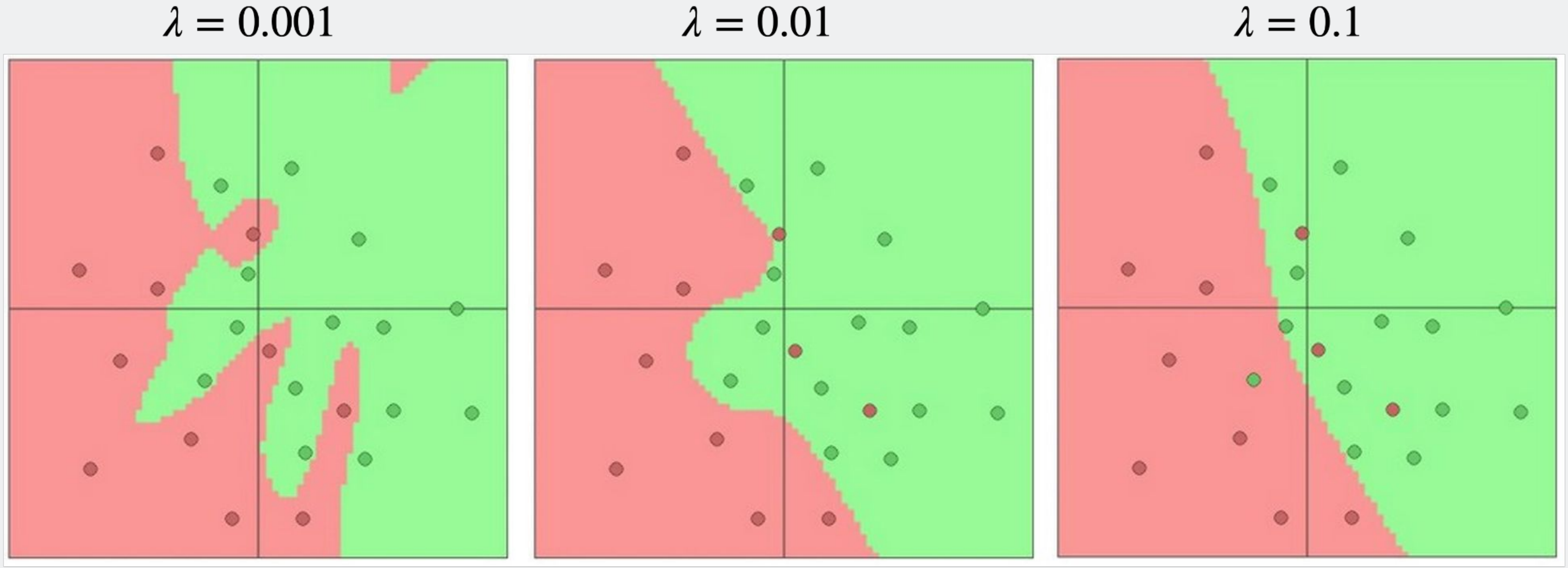
Points are linearly separable in feature space!

Setting the number of layers and their sizes



More hidden units = more capacity

Don’t regularize with size; instead use stronger L2



Web demo with ConvNetJS: https://cs.stanford.edu/people/karpathy/convnetjs/demo/classify2d.html

Universal Approximation

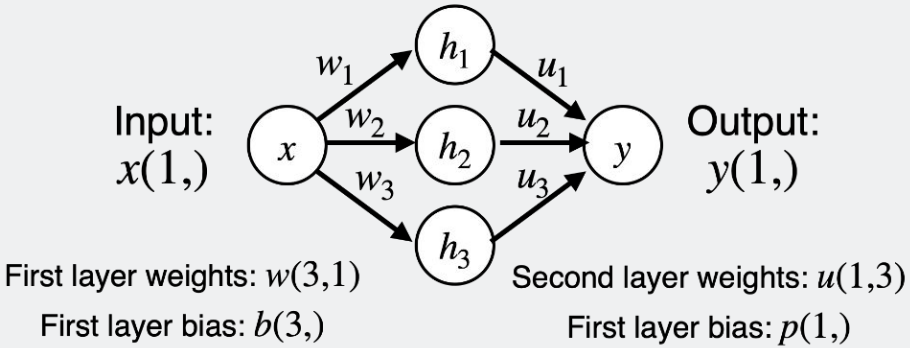
Universal Approximation

A neural network with one hidden layer can approximate any function $f : \mathbb{R}^N \to \mathbb{R}^M$ with arbitrary precision*

*Many technical conditions: Only holds on compact subsets of $\mathbb{R}^N$; function must be continuous; need to define "arbitrary precision"; etc.

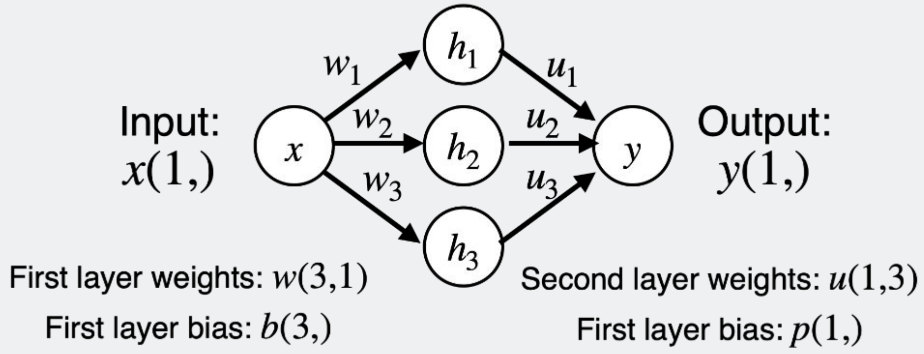
Universal Approximation

Example: Approximating a function $f: \mathbb{R} \rightarrow \mathbb{R}$ with a two-layer ReLU network



Universal Approximation

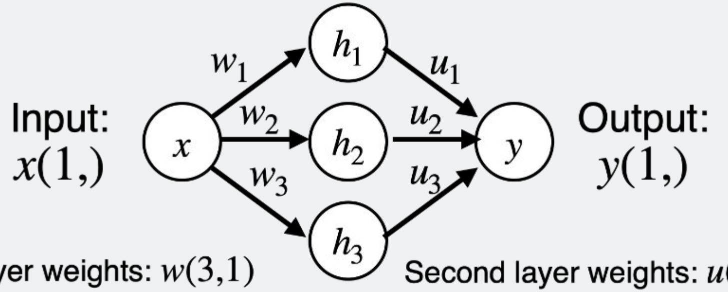
Example: Approximating a function $f : \mathbb{R} \to \mathbb{R}$ with a two-layer ReLU network



$$h_1 = \max(0, w_1 x + b_1)$$
$$h_2 = \max(0, w_2 x + b_2)$$
$$h_1 = \max(0, w_3 x + b_3)$$
$$y = u_1 h_1 + u_2 h_2 + u_3 h_3 + p$$

Universal Approximation

Example: Approximating a function $f : \mathbb{R} \rightarrow \mathbb{R}$ with a two-layer ReLU network



First layer weights: $w(3,1)$

First layer bias: $b(3,)$

Second layer weights: $u(1,3)$

First layer bias: $p(1,)$

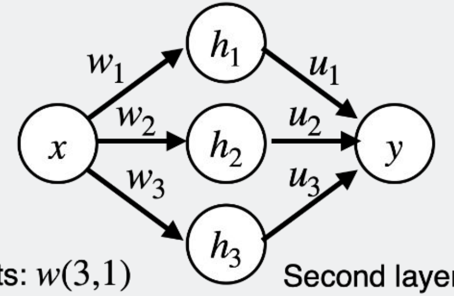
$$h_1 = \max(0, w_1 x + b_1)$$
$$h_2 = \max(0, w_2 x + b_2)$$
$$h_1 = \max(0, w_3 x + b_3)$$
$$y = u_1 h_1 + u_2 h_2 + u_3 h_3 + p$$

$$\begin{aligned} y = {} & u_1 \max(0, w_1 x + b_1) \\ & + u_2 \max(0, w_2 x + b_2) \\ & + u_3 \max(0, w_3 x + b_3) \\ & + p \end{aligned}$$

Universal Approximation

Example: Approximating a function $f : \mathbb{R} \rightarrow \mathbb{R}$ with a two-layer ReLU network

Input: $x(1,)$



Output: $y(1,)$

First layer weights: $w(3,1)$

First layer bias: $b(3,)$

Second layer weights: $u(1,3)$

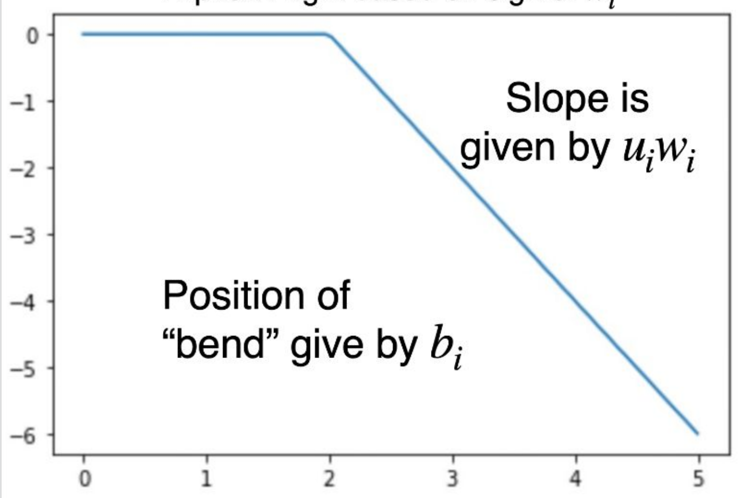
First layer bias: $p(1,)$

$$h_1 = \max(0, w_1 x + b_1)$$
$$h_2 = \max(0, w_2 x + b_2)$$
$$h_1 = \max(0, w_3 x + b_3)$$
$$y = u_1 h_1 + u_2 h_2 + u_3 h_3 + p$$

$$\begin{aligned} y = {} & u_1 \max(0, w_1 x + b_1) \\ & + u_2 \max(0, w_2 x + b_2) \\ & + u_3 \max(0, w_3 x + b_3) \\ & + p \end{aligned}$$

Output is a sum of shifted, scaled ReLUs:

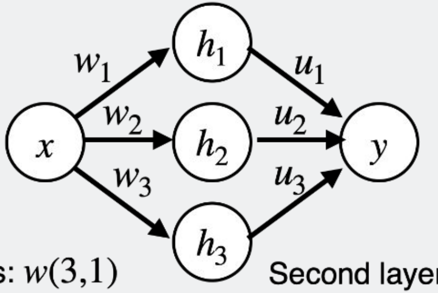
Flip left / right based on sign of w_i



Universal Approximation

Example: Approximating a function $f : \mathbb{R} \rightarrow \mathbb{R}$ with a two-layer ReLU network

Input: $x(1,)$



Output: $y(1,)$

First layer weights: $w(3,1)$

First layer bias: $b(3,)$

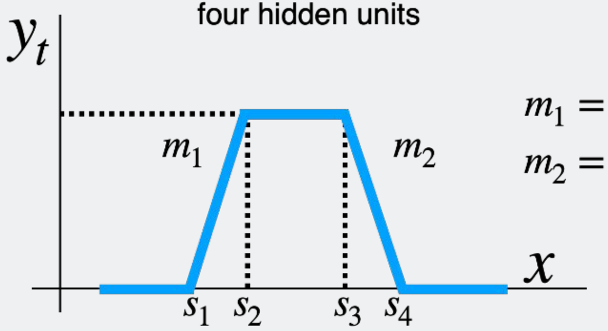
Second layer weights: $u(1,3)$

First layer bias: $p(1,)$

$$h_1 = \max(0, w_1 x + b_1)$$
$$h_2 = \max(0, w_2 x + b_2)$$
$$h_1 = \max(0, w_3 x + b_3)$$
$$y = u_1 h_1 + u_2 h_2 + u_3 h_3 + p$$

$$\begin{aligned} y = {} & u_1 \max(0, w_1 x + b_1) \\ & + u_2 \max(0, w_2 x + b_2) \\ & + u_3 \max(0, w_3 x + b_3) \\ & + p \end{aligned}$$

We can build a “bump function” using four hidden units

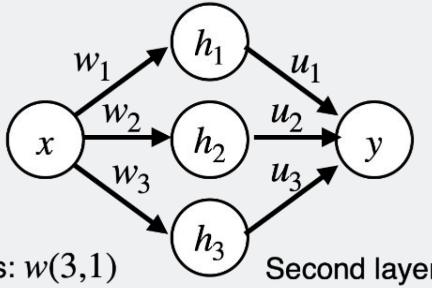


$$m_1 = t/(s_2 - s_1)$$
$$m_2 = t/(s_4 - s_3)$$

Universal Approximation

Example: Approximating a function $f : \mathbb{R} \rightarrow \mathbb{R}$ with a two-layer ReLU network

Input: $x(1,)$



Output: $y(1,)$

First layer weights: $w(3,1)$

First layer bias: $b(3,)$

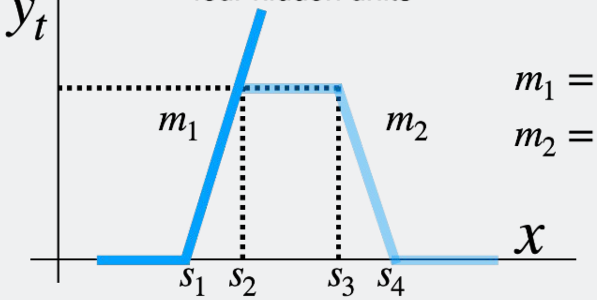
Second layer weights: $u(1,3)$

First layer bias: $p(1,)$

$$h_1 = \max(0, w_1 x + b_1)$$
$$h_2 = \max(0, w_2 x + b_2)$$
$$h_1 = \max(0, w_3 x + b_3)$$
$$y = u_1 h_1 + u_2 h_2 + u_3 h_3 + p$$

$$y = u_1 \max(0, w_1 x + b_1)$$
$$+ u_2 \max(0, w_2 x + b_2)$$
$$+ u_3 \max(0, w_3 x + b_3)$$
$$+ p$$

We can build a "bump function" using four hidden units



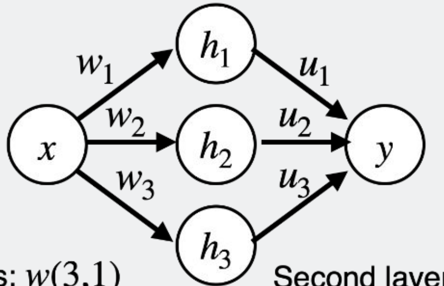
$$m_1 = t/(s_2 - s_1)$$
$$m_2 = t/(s_4 - s_3)$$

$m_1 \max(0, x - s_1)$

Universal Approximation

Example: Approximating a function $f: \mathbb{R} \rightarrow \mathbb{R}$ with a two-layer ReLU network

Input: $x(1,)$



Output: $y(1,)$

First layer weights: $w(3,1)$

First layer bias: $b(3,)$

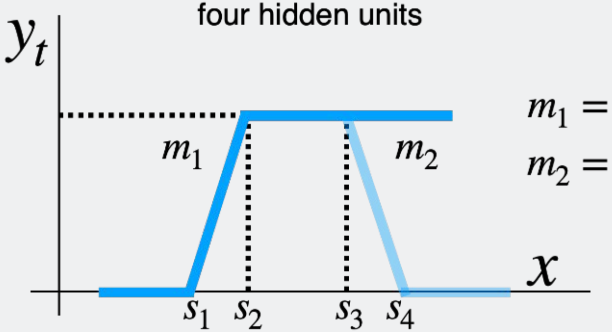
Second layer weights: $u(1,3)$

First layer bias: $p(1,)$

$$h_1 = \max(0, w_1 x + b_1)$$
$$h_2 = \max(0, w_2 x + b_2)$$
$$h_1 = \max(0, w_3 x + b_3)$$
$$y = u_1 h_1 + u_2 h_2 + u_3 h_3 + p$$

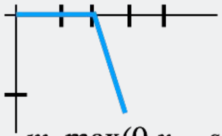
$$y = u_1 \max(0, w_1 x + b_1)$$
$$+ u_2 \max(0, w_2 x + b_2)$$
$$+ u_3 \max(0, w_3 x + b_3)$$
$$+ p$$

We can build a "bump function" using four hidden units



$$m_1 = t/(s_2 - s_1)$$
$$m_2 = t/(s_4 - s_3)$$

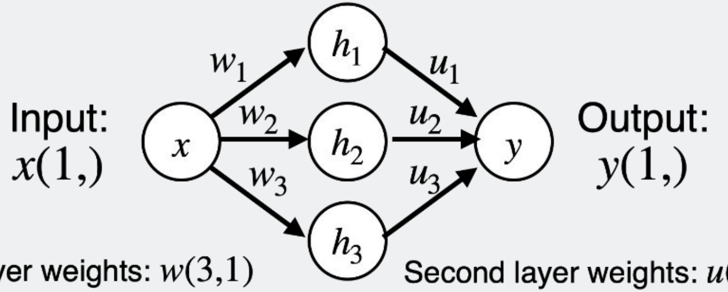
$m_1 \max(0, x - s_1)$



$-m_1 \max(0, x - s_2)$

Universal Approximation

Example: Approximating a function $f : \mathbb{R} \rightarrow \mathbb{R}$ with a two-layer ReLU network



First layer weights: $w(3,1)$

First layer bias: $b(3,)$

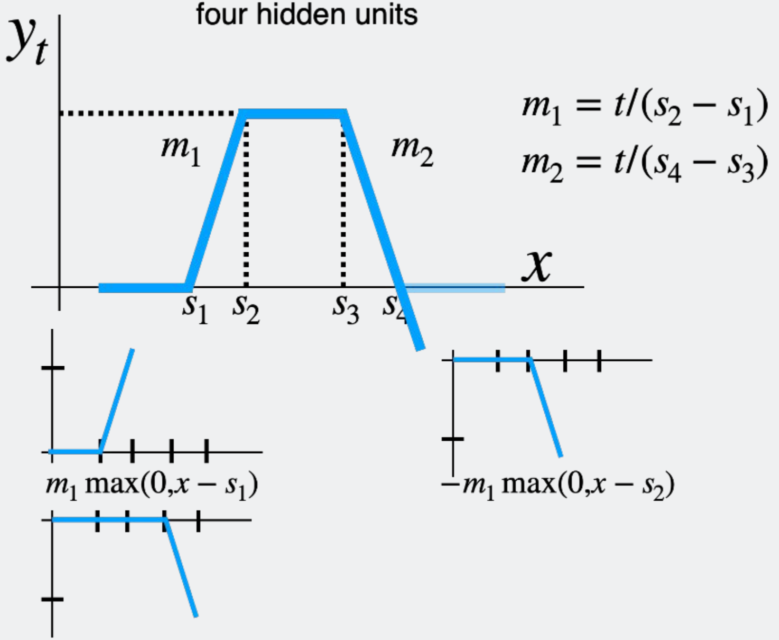
Second layer weights: $u(1,3)$

First layer bias: $p(1,)$

$$h_1 = \max(0, w_1 x + b_1)$$
$$h_2 = \max(0, w_2 x + b_2)$$
$$h_1 = \max(0, w_3 x + b_3)$$
$$y = u_1 h_1 + u_2 h_2 + u_3 h_3 + p$$

$$y = u_1 \max(0, w_1 x + b_1)$$
$$+ u_2 \max(0, w_2 x + b_2)$$
$$+ u_3 \max(0, w_3 x + b_3)$$
$$+ p$$

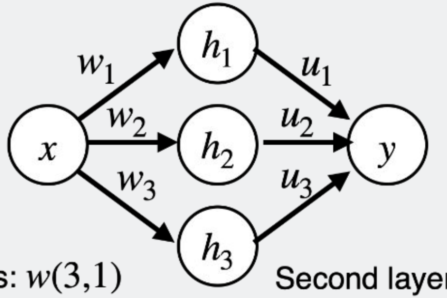
We can build a "bump function" using four hidden units



Universal Approximation

Example: Approximating a function $f : \mathbb{R} \rightarrow \mathbb{R}$ with a two-layer ReLU network

Input: $x(1,)$



Output: $y(1,)$

First layer weights: $w(3,1)$

First layer bias: $b(3,)$

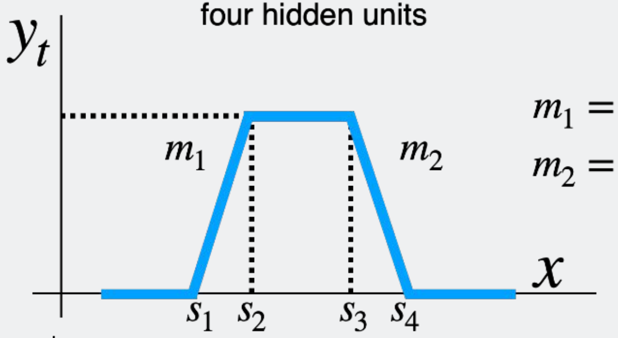
Second layer weights: $u(1,3)$

First layer bias: $p(1,)$

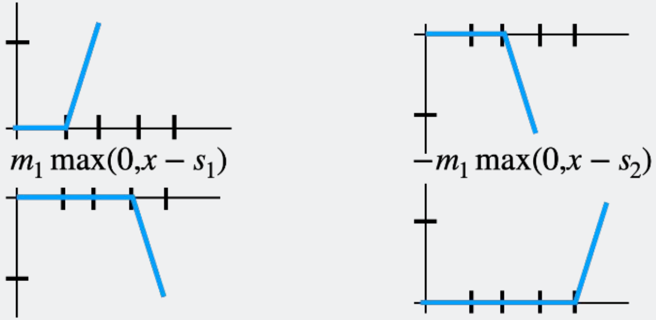
$$h_1 = \max(0, w_1 x + b_1)$$
$$h_2 = \max(0, w_2 x + b_2)$$
$$h_1 = \max(0, w_3 x + b_3)$$
$$y = u_1 h_1 + u_2 h_2 + u_3 h_3 + p$$

$$\begin{aligned} y = &\ u_1 \max(0, w_1 x + b_1) \\ &+ u_2 \max(0, w_2 x + b_2) \\ &+ u_3 \max(0, w_3 x + b_3) \\ &+ p \end{aligned}$$

We can build a "bump function" using four hidden units



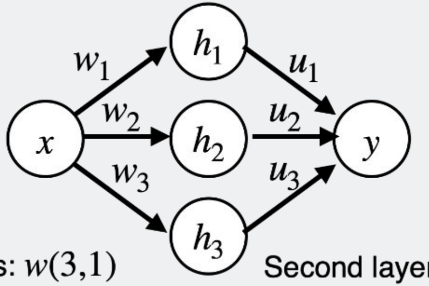
$$m_1 = t/(s_2 - s_1)$$
$$m_2 = t/(s_4 - s_3)$$



Universal Approximation

Example: Approximating a function $f : \mathbb{R} \rightarrow \mathbb{R}$ with a two-layer ReLU network

Input: $x(1,)$



Output: $y(1,)$

First layer weights: $w(3,1)$

First layer bias: $b(3,)$

Second layer weights: $u(1,3)$

First layer bias: $p(1,)$

$$h_1 = \max(0, w_1 x + b_1)$$
$$h_2 = \max(0, w_2 x + b_2)$$
$$h_1 = \max(0, w_3 x + b_3)$$
$$y = u_1 h_1 + u_2 h_2 + u_3 h_3 + p$$

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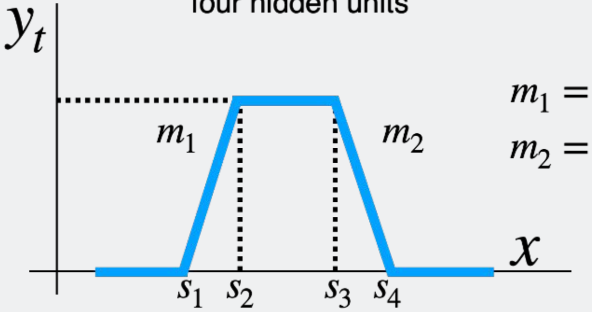
We can build a “bump function” using four hidden units



$$m_1 = t/(s_2 - s_1)$$
$$m_2 = t/(s_4 - s_3)$$

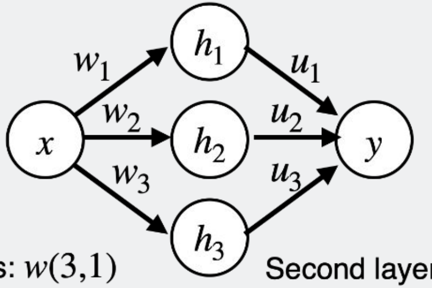
With 4K hidden units we can build a sum of K bumps

Approximate functions with bumps!

Universal Approximation

Example: Approximating a function $f : \mathbb{R} \to \mathbb{R}$ with a two-layer ReLU network

Input: $x(1,)$



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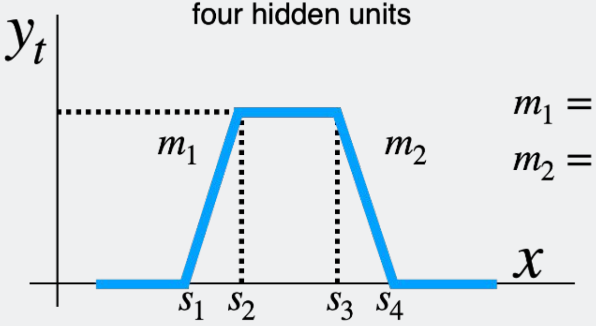
We can build a "bump function" using four hidden units



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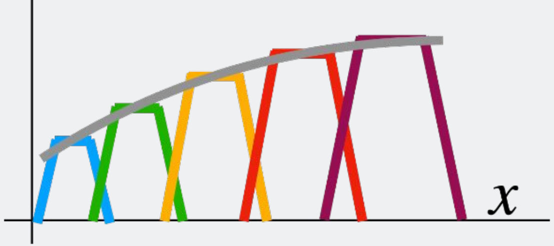
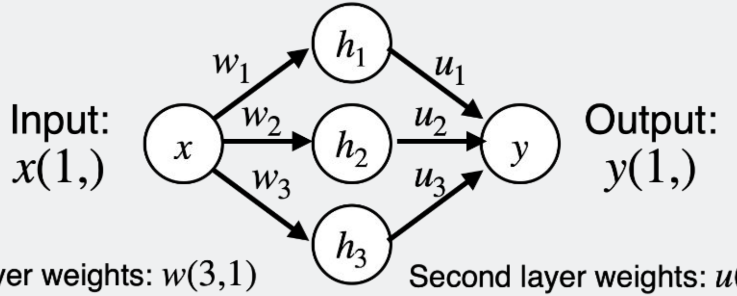


Approximate functions with bumps!

Universal Approximation

Example: Approximating a function $f : \mathbb{R} \to \mathbb{R}$ with a two-layer ReLU network



First layer weights: $w(3,1)$

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$$y = u_1 \max(0, w_1 x + b_1) + u_2 \max(0, w_2 x + b_2) + u_3 \max(0, w_3 x + b_3) + p$$

What about …

- Gaps between bumps?
- Other nonlinearities?
- Higher-dimensional functions?

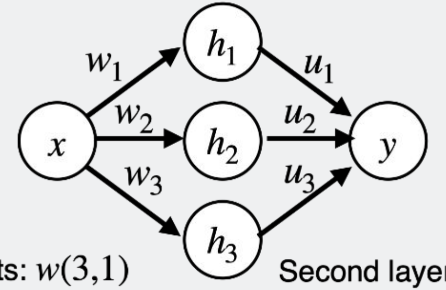
See [Nielsen, Chapter 4](#)

Approximate functions with bumps!

Universal Approximation

Example: Approximating a function $f : \mathbb{R} \rightarrow \mathbb{R}$ with a two-layer ReLU network

Input: $x(1,)$



Output: $y(1,)$

First layer weights: $w(3,1)$

First layer bias: $b(3,)$

Second layer weights: $u(1,3)$

First layer bias: $p(1,)$

$$h_1 = \max(0, w_1 x + b_1)$$
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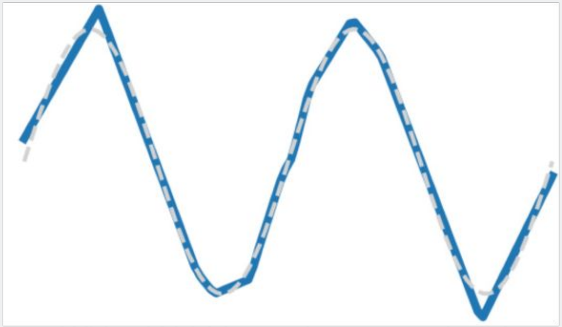
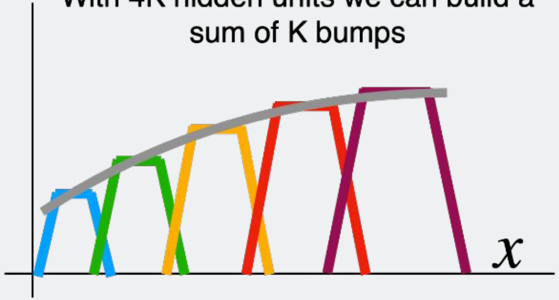
$$y = u_1 \max(0, w_1 x + b_1) + u_2 \max(0, w_2 x + b_2) + u_3 \max(0, w_3 x + b_3) + p$$

Reality check: Networks don't really learn bumps!



With 4K hidden units we can build a sum of K bumps

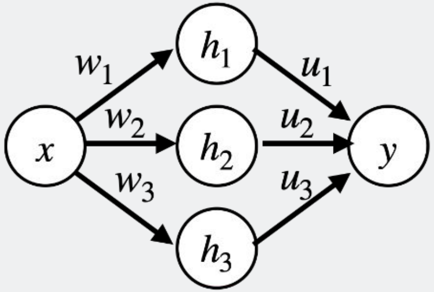


Approximate functions with bumps!

Universal Approximation

Example: Approximating a function $f : \mathbb{R} \rightarrow \mathbb{R}$ with a two-layer ReLU network

Input: $x(1,)$



Output: $y(1,)$

Universal approximation tells us:

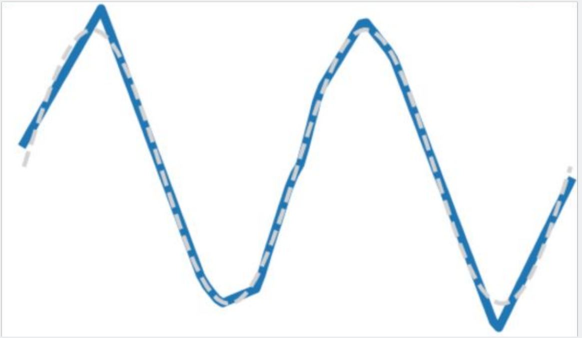
- Neural nets can represent any function

Universal approximation DOES NOT tell us:

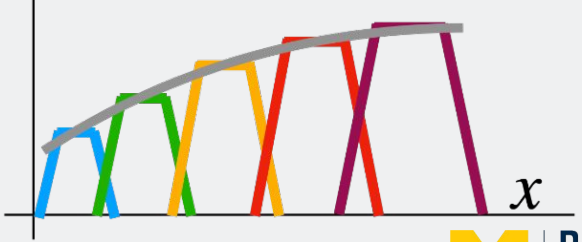
- Whether we can actually learn any function with SGD
- How much data we need to learn a function

Remember: kNN is also a universal approximator!

Reality check: Networks don't really learn bumps!



With 4K hidden units we can build a sum of K bumps



Approximate functions with bumps!

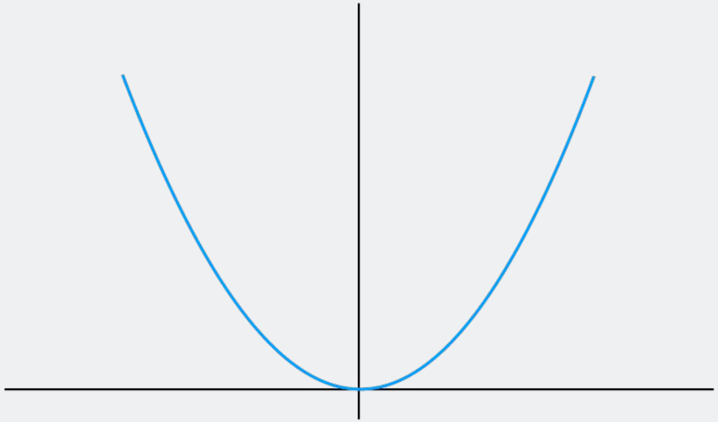
Convex Functions

Convex Functions

A function $f : X \subseteq \mathbb{R}^N \to \mathbb{R}$ is **convex** if for all $x_1, x_2 \in X, t \in [0,1]$,

$$f(tx_1 + (1-t)x_2 \leq tf(x_1) + (1-t)f(x_2)$$

Example: $f(x) = x^2$ is convex:



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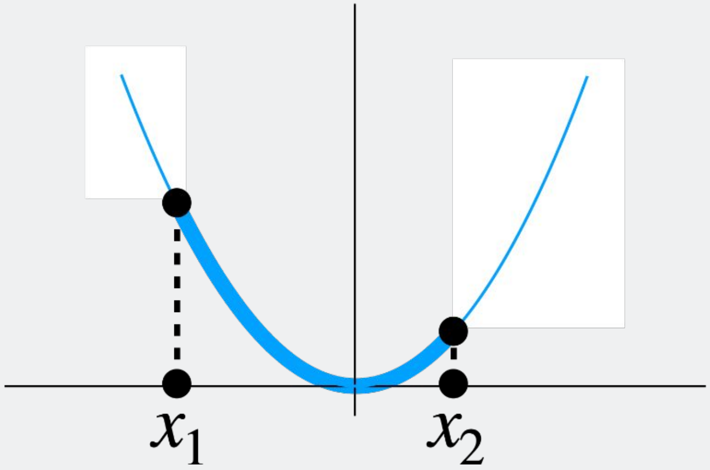

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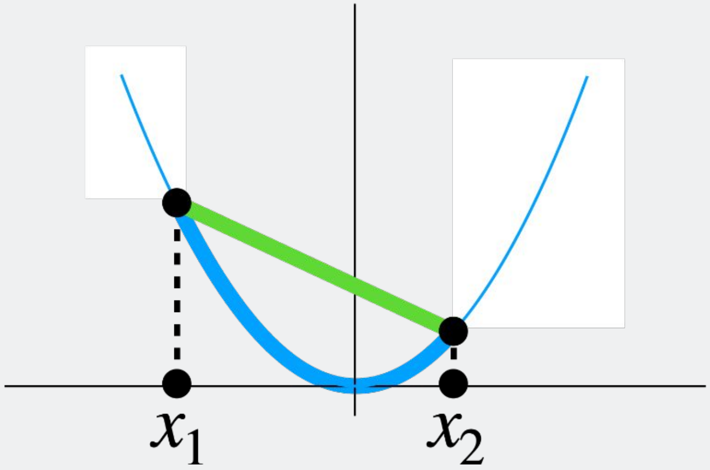

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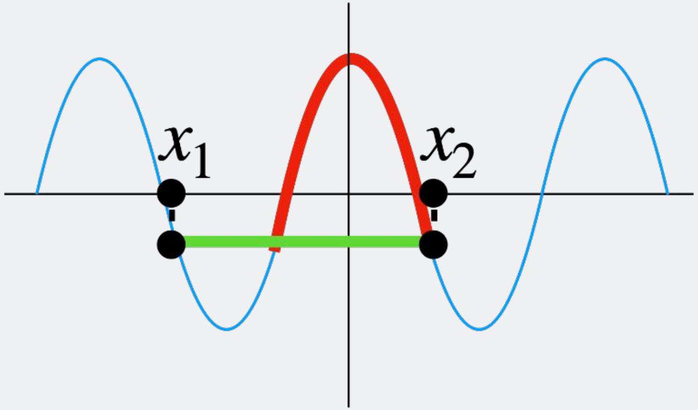
Example: $f(x) = \cos(x)$ is not convex:



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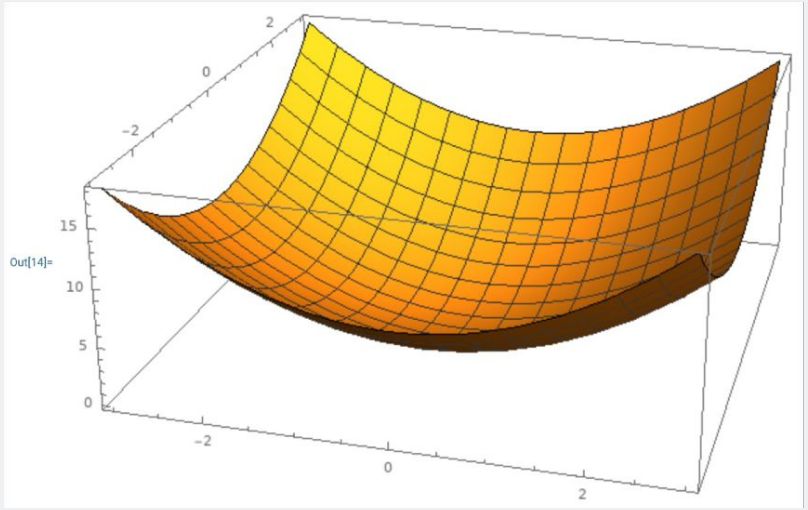
Intuition: A convex function is a (multidimensional) bowl

Generally speaking, convex functions are **easy to optimize**: can derive theoretical guarantees about **converging to global minimum***



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Linear classifiers optimize a **convex function!**

$$s = f(x; W) = Wx$$

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum + je^{s_j}}\right)$$ Softmax

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$ SVM

$$L = \frac{1}{N}\sum_{i=1}^{N} L_i + R(W)$$ where $R(W)$ is L2 or L1 regularization

Convex Functions

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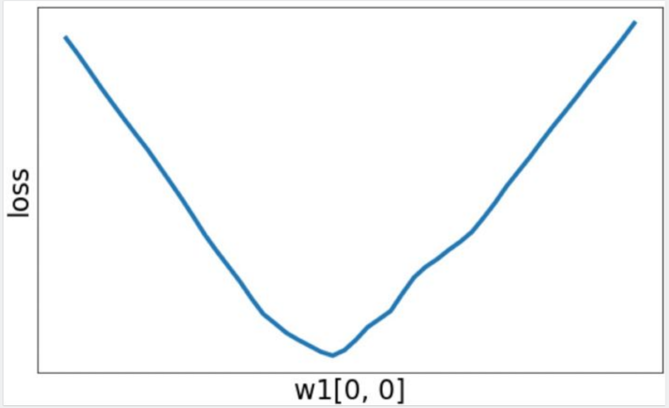
Neural net losses sometimes look convex-ish:



ROBOTICS

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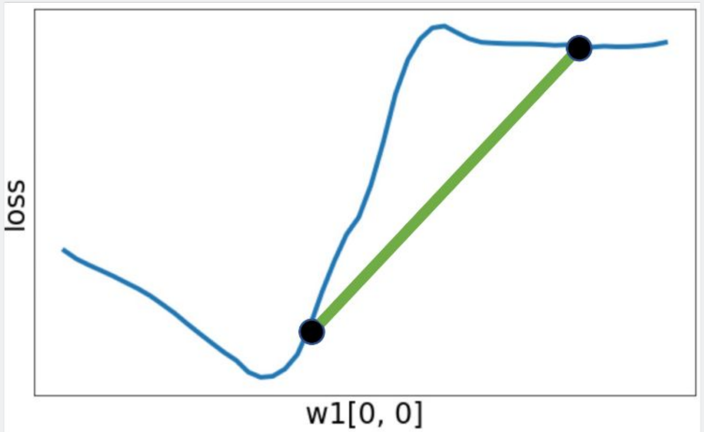
Generally speaking, convex functions are **easy to optimize**: can derive theoretical guarantees about **converging to global minimum***

But often clearly nonconvex:



1D slice of loss landscape for a 4-layer ReLU network with 10 input features, 32 units per hidden layer, 10 categories, with softmax loss

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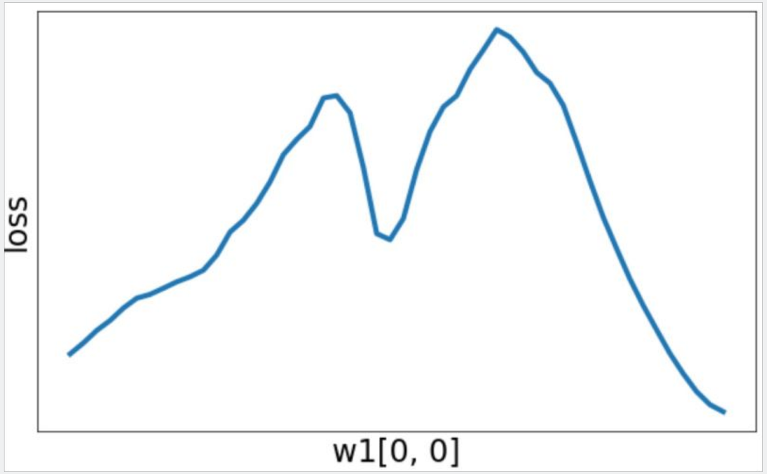
With local minima:



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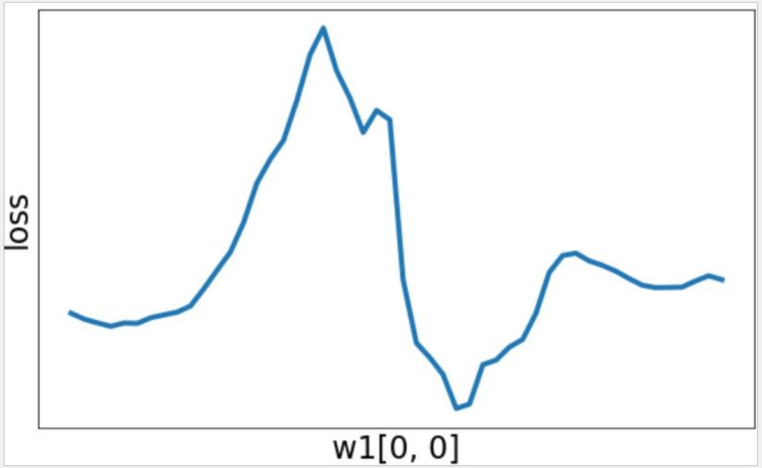
Can get very wild!



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Intuition: A convex function is a (multidimensional) bowl

Generally speaking, convex functions are **easy to optimize**: can derive theoretical guarantees about **converging to global minimum***

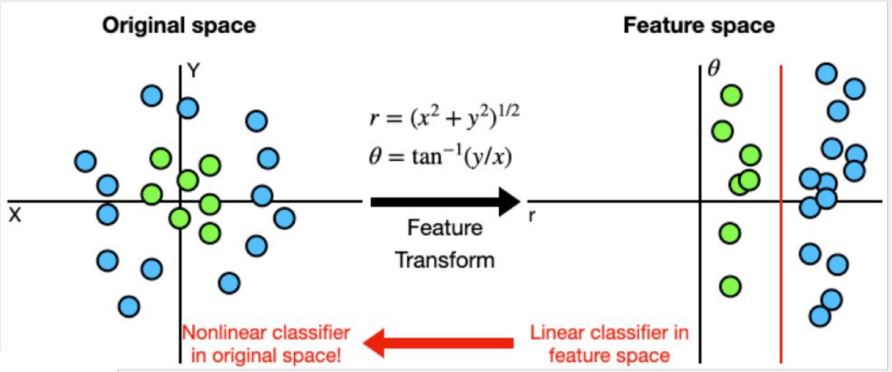
Most neural networks need **nonconvex optimization**

- Few or no guarantees about convergence
- Empirically it seems to work anyway
- Active area of research

Summary

Summary

Feature transform + Linear classifier allows nonlinear decision boundaries

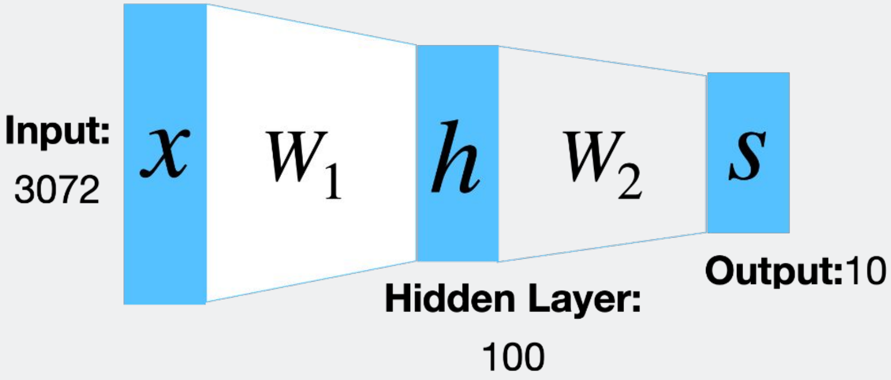


Neural Networks as learnable feature transforms

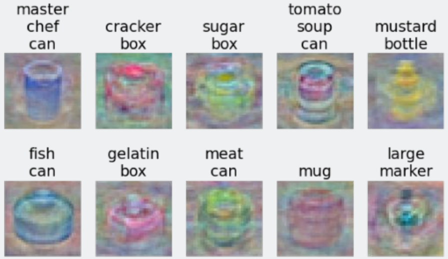
Summary

From linear classifiers to fully-connected networks

$$f(x) = W_2 \max(0, W_1 x + b_1) + b_2$$



Linear classifier: One template per class



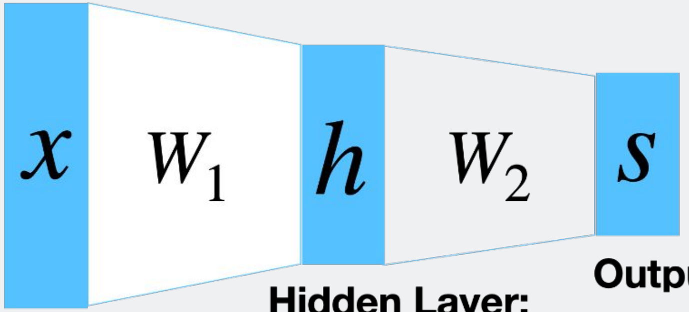
Neural networks: Many reusable templates

Summary

From linear classifiers to
fully-connected networks

$$f(x) = W_2 \max(0, W_1 x + b_1) + b_2$$

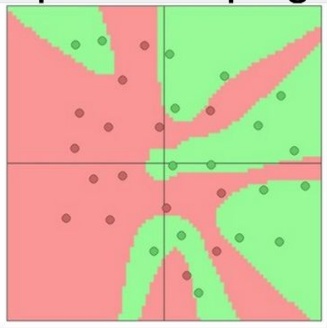
Input: 3072



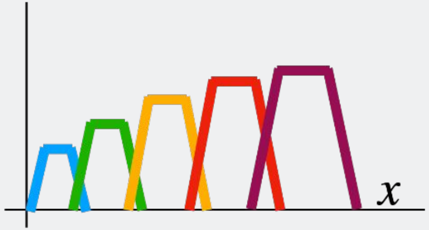
Hidden Layer: 100

Output:10

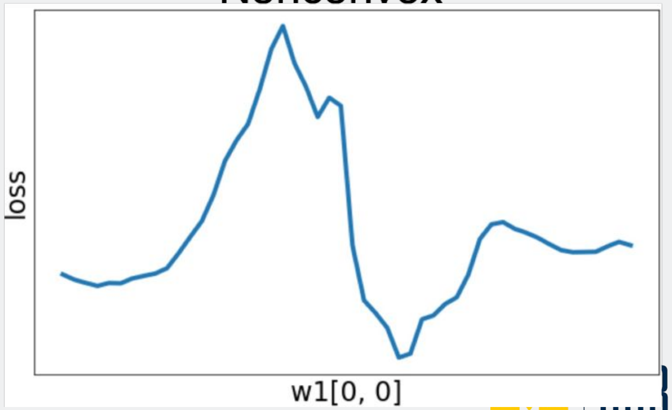
Space Warping



Universal approximation



Nonconvex



Problem: How to compute gradients?

$$s = W_2 \max(0, W_1 x + b_!) + b_2$$ Nonlinear score function

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$ Per-element data loss

$$R(W) = \sum_k W_k^2$$ L2 regularization

$$L(W_1, W_2, b_1, b_2) = \frac{1}{N} \sum_{i=1}^{N} L_i + \lambda R(W_1) + \lambda R(W_2)$$ Total loss

If we can compute $\frac{\delta L}{\delta W_1}, \frac{\delta L}{\delta W_2}, \frac{\delta L}{\delta b_1}, \frac{\delta L}{\delta b_2}$ then we can optimize with SGD

Next up: Backpropagation